Classical Mechanics I

Homework Set 3

1. A simple harmonic oscillator consists of a 100 *g* mass attached to a spring whose force constant k is 10 *N*/*m*. The mass is displaced 3 *cm* and released from rest. Calculate
   - *a.* The natural frequency f_o and the natural period T_o
   - *b.* The total energy of the system (*in Joules*)
   - *c.* The maximum speed of the mass (*in cm*/*s*)

2. A guitar string vibrates harmonically with a frequency of 512 Hz. If the amplitude of oscillation of the centerpoint of the string is 0.002 *m* and $x(0) = 0.002$ *m*,
   - *a.* Find the maximum speed of the centerpoint
   - *b.* The maximum acceleration of the centerpoint

3. One the surface of the Moon, the acceleration of gravity is about one-sixth that on the Earth. Calculate the period of oscillation of a 1 *m* long pendulum in both locations and compare.

4. A simple harmonic oscillator consists of a 100 *g* mass attached to a spring whose force constant k is 10 *N*/*m*. The mass is displaced 3 *cm* and released from rest in a resistive medium. After oscillating for 10 s, the maximum amplitude decreases to half the initial value. Assuming the motion is underdamped, calculate
   - *a.* The damping parameter β
   - *b.* The frequency f and then compare it to the un-damped frequency found in Prob. 1, Part *a*.
     Hint: It will prove helpful to take your answer to 6 decimal places
   - *c.* The decrement of the motion, R

5. **Suppose the motion of a particle is described by $\ddot{x} - 2\beta\dot{x} + \omega_o^2 x = 0$, where the damping resistance of the medium is ***negative***. In this case, the general solution is

   $$x(t) = e^{\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_o^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_o^2}\,t}\right].$$

   Determine expressions for the general solution when $\beta^2 > \omega_o^2$, $\beta^2 = \omega_o^2$, and $\beta^2 < \omega_o^2$. Discuss the motion of the particle in each case.

6. A system is undergoing simple harmonic motion with the initial position x_o and velocity v_o at $t = 0$. If we assume the solution has the form

   $$x(t) = B_1 \cos(\omega_o t) + B_2 \sin(\omega_o t)$$

   - *a.* If the oscillator's mass is $m = 0.5$ *kg* and the force constant $k = 50$ N/m, find the angular frequency ω_o.
   - *b.* Find B_1 and B_2 for $x_o = 3.0$ *m* and $v_o = 50$ *m*/*s*. Use your value of ω_o from part (*a*).

7. Consider a damped harmonic oscillator undergoing underdamped motion. After four cycles, the amplitude of the oscillator has dropped to $1/e$ of its initial value. Find ω/ω_o, the ratio of the frequency of the damped oscillator to its natural frequency out to 5 decimal places.

8. **To better understand underdamped motion, use a computer to plot $x(t)$ for

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$$

along with its two components

$$x_E(t) = \pm Ae^{-\beta t}$$ Amplitude Limits (*Envelope*)

$$x_{SHM}(t) = A\cos(\omega_1 t - \delta)$$ Simple Harmonic Motion (*no damping*)

on the same graph. Let $A = 1\ m$ and assume the damping is small so that $\omega_1 \approx \omega_o = 1\ rad/s$. Plot this for each of the NINE possible combinations shown below and discuss the results. Set the *amplitude axis* limits to ±1 and the *time axis* limits from 0 to 15 *sec*.

β^2/ω_o^2	δ		β^2/ω_o^2	δ		β^2/ω_o^2	δ
0.1	0, π/2 , π		0.5	0, π/2 , π		0.9	0, π/2 , π

9. **Use the general solutions for $x(t)$ to the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0$$

for underdamped, critically damped, and overdamped motion, along with the following initial conditions, to find the appropriate constants in each case:

$$x(0) = x_o \qquad \text{and} \qquad v(0) = v_o = 0$$

Use a computer to plot the results for $x(t)/x_o$ as a function of $\omega_o t$ in all three cases with $\beta = (1/2)\omega_o$ for underdamped, $\beta = \omega_o$ for critically damped, and $\beta = 2\,\omega_o$ for overdamped. Show all three plots on the same graph for comparison. Set the *amplitude axis* limits from +1 to -0.25 and the *time axis* limits from 0 to 25 *sec*.

10. **The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_o\left(\frac{r}{R} + \lambda^2\frac{R}{r}\right) \qquad \textit{for } 0 < r < \infty$$

with U_o, R, and λ all positive constants.

a. Find the equilibrium position r_o

b. Using your answer for r_o from part (*a*), let x be the distance measured from equilibrium, such that $r = r_o + x$. Show that for small x, $U(r)$ has the form

$$U = const + \frac{1}{2}kx^2 \qquad \textit{with } k = \textit{collection of constants}$$

Hint: After subbing in for r, make use of the power series expansion $(1+\beta)^{-1} \approx 1 - \beta + \beta^2$

c. What is the angular frequency for small oscillations about r_o?