Vertical Motion - Multi-Stage Rockets

Single-Stage rockets would require a large amount of fuel to put a small payload into a low earth orbit (LEO) even if the effects of gravity AND air resistance are ignored. Neglecting air resistance is not a bad approximation in many cases as careful rocket design can minimize its effects. However, the effects of gravity can not be ignored. One way to counter its effects is to use multi-stage rockets.

How much fuel is needed to boost a rocket and payload into LEO with a final velocity of 8 km/s?

Single-Stage Rocket:

Equation of motion of a rocket moving vertical under the influence of gravity is given by

$$m\frac{d\mathbf{v}}{dt} - \boldsymbol{\mu}\frac{dm}{dt} = m\mathbf{g} \, . \tag{1.1}$$

If we choose up as positive, after a little rearranging we get:

$$\frac{dv}{\mu} = -\frac{dm}{m} - \frac{g}{\mu}dt \, . \tag{1.2}$$

For the rocket to achieve liftoff, the first term on the right hand side (RHS) must exceed the second. The reciprocal of the constant $\frac{g}{\mu}$ is considered a "parameter of goodness" which is unique to every rocket and is given a special name, the ***specific impulse***, $\boldsymbol{\tau_s}$, of the rocket engine.

$$\rightarrow \qquad \tau_s = \frac{\mu}{g}$$

NOTE: τ_s has units of time (*s*), and its value depends on the exhaust velocity (μ) of the rocket. Thus, it depends primarily on the thermodynamics of what goes on inside the rocket's combustion chamber and the shape of the engine nozzle. A well-designed chemical rocket has an exhaust velocity of about 3000 *m*/*s* with an average molecular weight of the combustibles of about 30.

$$\rightarrow \qquad \boldsymbol{\tau_s} = \frac{\mu}{g} = \frac{3000}{9.8} \approx 300 \, s \; .$$

The velocity of the rocket as a function of time can be found by integrating equation 1.2,

$$\frac{1}{\mu}\int_{0}^{v} dv = -\int_{m_o}^{m}\frac{dm}{m} - \frac{1}{\tau_s}\int_{0}^{t} dt \tag{1.3}$$

which yields:

$$\frac{v}{\mu} = \ln\left(\frac{m_o}{m}\right) - \frac{t}{\tau_s} \tag{1.4}$$

or

$$\frac{v}{\mu} = \ln\left(\frac{m_R + m_p + m_F}{m_R + m_p + m'_F}\right) - \frac{t}{\tau_s} \tag{1.5}$$

with m_R = rocket mass, m_p = payload mass, m_F = initial fuel mass, and m'_F = the current fuel mass.

** At burnout, $t = \tau_b$, and $m'_F = 0$.

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$$\frac{v}{\mu} = \ln\left(\frac{m_R + m_p + m_F}{m_R + m_p}\right) - \frac{\tau_b}{\tau_s} \tag{1.6}$$

Solving for the mass ratio yields:

$$\left(\frac{m_R + m_p + m_F}{m_R + m_p}\right) = e^{\left(\frac{v}{\mu} + \frac{\tau_b}{\tau_s}\right)} \tag{1.7}$$

Since we are looking for how much fuel is need to put a rocket and payload into LEO, we now solve 1.7 for the mass of fuel relative to the mass of the rocket + payload:

$$\left(\frac{m_F}{m_R + m_p}\right) = e^{\left(\frac{v}{\mu} + \frac{\tau_b}{\tau_s}\right)} - \left(\frac{m_R}{m_R + m_p}\right) - \left(\frac{m_p}{m_R + m_p}\right) \tag{1.8}$$

Since $m_R >> m_p$, we can simply 1.8 down to:

$$\left(\frac{m_F}{m_R + m_p}\right) \approx e^{\left(\frac{v}{\mu} + \frac{\tau_b}{\tau_s}\right)} - 1 \tag{1.9}$$

** Assuming the burnout time of a particular rocket lifting a payload into LEO (v = 8000 *m*/*s*) is about 600 *s*, with a specific impulse of 300 *s*, and an exhaust velocity of 3000 *m*/*s*, the mass ratio becomes:

$$\left(\frac{m_F}{m_R+m_p}\right)\approx e^{(2.67+2.00)}-1=105$$

In other words, it takes about 105 *kg* of fuel to place 1 *kg* of *stuff* into orbit!

Multi-Stage Rocket:

Multi-Stage rockets are more efficient for putting payloads into LEO. The tanks that hold the fuel for the first stage are jettisoned after the first stage burnout. This excess/useless mass is now no longer boosted into orbit, which greatly reduces the overall fuel requirement.

Define the mass ratio α as

$$\alpha=\frac{m_R+m_p+m_F}{m_R+m_p}\,. \tag{1.10}$$

Assuming that the mass ratio of the first stage α_1 is equal to that of the second α_2 and the burnout times for each stage τ_{b1} and τ_{b2} are identical, the final velocities of each stage can be found by integrating equation 1.3 :

Stage I $$\frac{v_{1f}}{\mu}=\ln\alpha-\frac{\tau_b}{\tau_s} \tag{1.11}$$

Stage II $$\frac{v_{2f}-v_{1f}}{\mu}=\ln\alpha-\frac{\tau_b}{\tau_s} \tag{1.12}$$

with $\alpha_1=\alpha_2=\alpha$ and $\tau_{b1}=\tau_{b2}=\tau_b$.

Combing the previous two expression and solving for v_{2f} gives:

$$\frac{v_{2f}}{\mu}=2\ln\alpha-2\frac{\tau_b}{\tau_s} \tag{1.13}$$

Again solving for the fuel to rocket + payload mass ratio as before yields:

$$\left(\frac{m_F}{m_R+m_p}\right)\approx e^{\left(\frac{v_{2f}}{2\mu}+\frac{\tau_b}{\tau_s}\right)}-1 \tag{1.14}$$

Putting in the same numbers used previously we get

$$\left(\frac{m_F}{m_R + m_p}\right) \approx e^{(1.33+2.00)} - 1 = 27$$

Thus, it takes only about 27 *kg* of fuel to put 1 *kg* of *stuff* into LEO using a two stage rocket!

Ex. **Titan II**

Size Height 31.4 *m*
Diameter 3.05 *m*
Mass 154,000 *kg*
Stages 2
Payload Capacity to LEO 3,600 *kg*
Payload to Polar LEO 2,177 *kg*
Payload to Escape 227 *kg* (500 *lb*)

First Stage

Thrust 1,900 *kN*
Specific impulse 258 *s*
Burn time 156 *s*

Second Stage

Thrust 445 *kN*
Specific impulse 316 *s*
Burn time 180 *s*