Ex. Convert the following vector from Cartesian to Spherical Coordinates and verify that its *magnitude* is the same in both systems.

$$\mathbf{r} = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}} + 5\hat{\mathbf{z}}$$

Recall that the representation of a point P in a particular system is given by just listing the 3 corresponding coordinates in triplet form:

(x, y, z)	Cartesian
(r, θ, φ)	Spherical

and that we could convert the point P's location from one coordinate system to another using coordinate transformations.

Cartesian → Spherical

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

Spherical → Cartesian

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

To convert the vector from Cartesian to Spherical Coordinates, we must convert the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ unit vectors in to $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ unit vectors using:

$$\hat{\mathbf{x}} = \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$

The expression that needs to be evaluated looks like this:

$$\begin{aligned}\mathbf{r} = 3&\left(\sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}}\right)\\ &+ 4\left(\sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}}\right)\\ &\quad + 5\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right)\end{aligned}$$

Before **r** can be determined, θ and φ must be found.

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{3^2+4^2}}{5}\right) = \frac{\pi}{4} \quad (45^\circ)$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = .9272952 \quad (53.13^\circ)$$

Rearranging **r** to group like terms together yields:

$$\begin{aligned}\mathbf{r} = &\left(3\sin\theta\cos\varphi + 4\sin\theta\sin\varphi + 5\cos\theta\right)\hat{\mathbf{r}} \\ &+ \left(3\cos\theta\cos\varphi + 4\cos\theta\sin\varphi - 5\sin\theta\right)\hat{\boldsymbol{\theta}} \\ &+ \left(-3\sin\varphi + 4\cos\varphi\right)\hat{\boldsymbol{\varphi}}\end{aligned}$$

Substituting our values for θ and φ, we get:

$$\begin{aligned}\mathbf{r} = &\left(3\sin\left(\frac{\pi}{4}\right)\cos(.9272952) + 4\sin\left(\frac{\pi}{4}\right)\sin(.9272952) + 5\cos\left(\frac{\pi}{4}\right)\right)\hat{\mathbf{r}} \\ &+ \left(3\cos\left(\frac{\pi}{4}\right)\cos(.9272952) + 4\cos\left(\frac{\pi}{4}\right)\sin(.9272952) - 5\sin\left(\frac{\pi}{4}\right)\right)\hat{\boldsymbol{\theta}} \\ &+ \left(-3\sin(.9272952) + 4\cos(.9272952)\right)\hat{\boldsymbol{\varphi}}\end{aligned}$$

Reducing yields:

$$\begin{aligned}\mathbf{r} = &\left(3\left(\frac{\sqrt{2}}{2}\right)(.6) + 4\left(\frac{\sqrt{2}}{2}\right)(.8) + 5\left(\frac{\sqrt{2}}{2}\right)\right)\hat{\mathbf{r}} \\ &+ \left(3\left(\frac{\sqrt{2}}{2}\right)(.6) + 4\left(\frac{\sqrt{2}}{2}\right)(.8) - 5\left(\frac{\sqrt{2}}{2}\right)\right)\hat{\boldsymbol{\theta}} \\ &+ \left(-3(.8) + 4(.6)\right)\hat{\boldsymbol{\varphi}}\end{aligned}$$

$$\mathbf{r} = \left(5\sqrt{2}\right)\hat{\mathbf{r}} \; + \; (0)\hat{\boldsymbol{\theta}} \; + \; (0)\hat{\boldsymbol{\varphi}}$$

$$\mathbf{r} = \left(5\sqrt{2}\right)\hat{\mathbf{r}}$$

Therefore, the equivalent (*but not unique*) vector in spherical coordinates is:

$$\mathbf{r} = \left(5\sqrt{2}\right)\hat{\mathbf{r}} \qquad \rightarrow$$

$\mathbf{r}$ will only be unique if θ and φ are given. Otherwise, $\mathbf{r}$ would map out the surface of a sphere.

Magnitudes:

In Cartesian coordinates, the magnitude of **r** is given by:

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

➔ $$|\mathbf{r}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$$

In Spherical coordinates, the magnitude of **r** is given by:

$$|\mathbf{r}| = r$$

➔ $$|\mathbf{r}| = 5\sqrt{2}$$