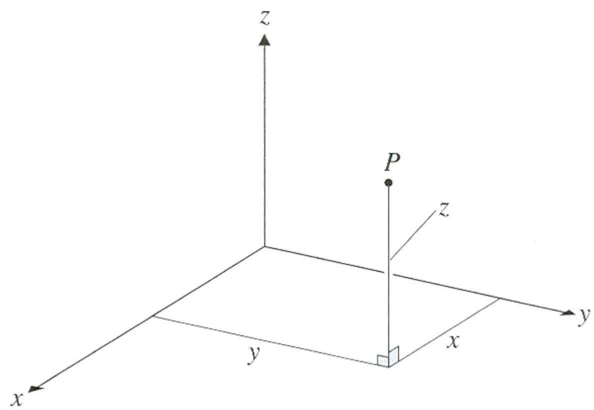
Properties of Coordinate Systems

Cartesian Coordinates



Variable Limits:

$-\infty < x < \infty$

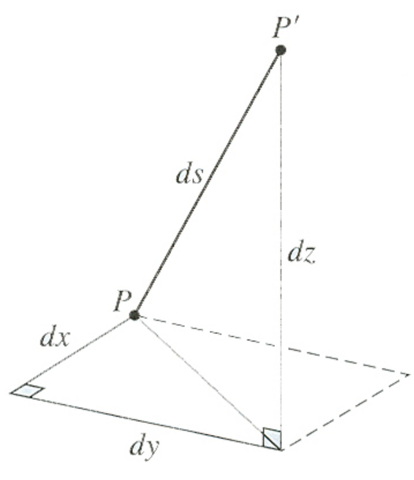
$-\infty < y < \infty$

$-\infty < z < \infty$

Position vector:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

For Two Neighboring Points P and P':



Displacement between two neighboring points:

$$d\mathbf{s} = d\mathbf{r} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

Distance between two neighboring points (*Found using the Pythagorean Theorem*):

$$ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2 + dz^2}$$

Primary Curve – the curve obtained when one coordinate variable is allowed to vary while the other two are held fixed.

Primary Length Element – infinitesimal length along the primary curve

Primary Surface – the surface obtained when the coordinate determining the primary length element is held fixed and the other two are allowed to vary.

Primary Element	Primary Curve	Primary Surface	Primary Volume
1st : x	Straight Line (*x-axis*) (y and z fixed, x varies)	*yz*-plane (x fixed , y and z varies)	
2nd : y	Straight Line (*y-axis*) (x and z fixed, y varies)	*xz*-plane (y fixed , x and z varies)	Solid Cube
3rd : z	Straight Line (*z-axis*) (x and y fixed, z varies)	*xy*-plane (z fixed , x and y varies)	

Primary Length Elements (dl)	Primary Area Elements (dA)	Primary Volume Elements (dV)
1st: dx $(\hat{\mathbf{x}})$	$dy\,dz$ $(\hat{\mathbf{x}})$	
2nd: dy $(\hat{\mathbf{y}})$	$dx\,dz$ $(\hat{\mathbf{y}})$	$dx\,dy\,dz$
3rd: dz $(\hat{\mathbf{z}})$	$dx\,dy$ $(\hat{\mathbf{z}})$	

Primary length element vectors are in the direction of their corresponding primary curve.

Primary area element vectors are in the same direction as the primary length element vector (*i.e.* $\perp$ *to their corresponding primary surface*).

Primary volume elements are scalars not vectors and do not have an associated direction.

Coordinate Conversions

Spherical → Cartesian

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Cylindrical → Cartesian

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$
$$z_{Cart} = z_{cyl}$$

Unit Vector Conversions

Spherical → Cartesian

$$\hat{\mathbf{r}} = \sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$

Cylindrical → Cartesian

$$\hat{\boldsymbol{\rho}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}}_{cyl} = \hat{\mathbf{z}}_{Cart}$$

Special Functions Involving the Del Operator (∇)

Gradient:

$$\vec{\nabla} f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

Divergence:

$$\vec{\nabla}\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Curl:

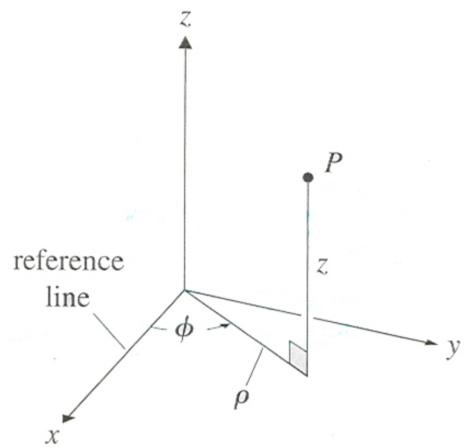
$$\vec{\nabla}\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Note: *f could also be a vector function* **F**

Cylindrical Coordinates



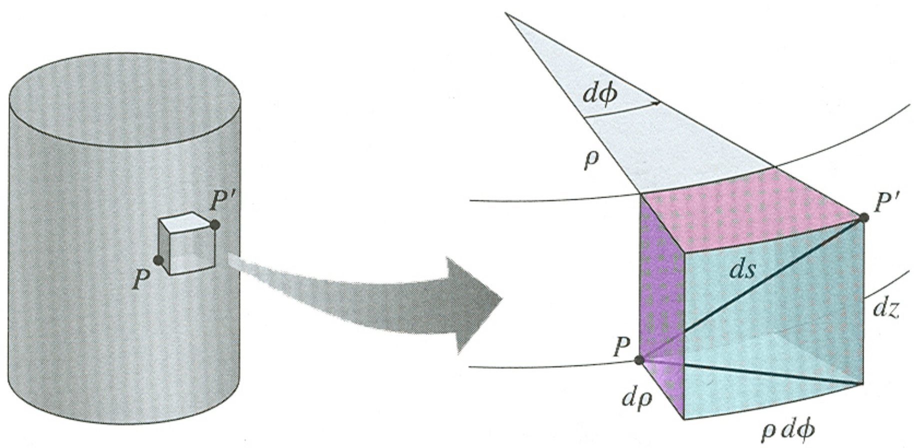
Variable Limits:

$$0 \le \rho < \infty$$
$$0 \le \phi \le 2\pi$$
$$-\infty < z < \infty$$

Position vector:

$$\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}} \qquad or \qquad \mathbf{r} = \rho\cos\varphi\,\hat{\mathbf{x}} + \rho\sin\varphi\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

For Two Neighboring Points P and P':



Displacement between two neighboring points:

$$d\mathbf{s} = d\mathbf{r} = d\rho\,\hat{\boldsymbol{\rho}} + \rho d\varphi\,\hat{\boldsymbol{\varphi}} + dz\,\hat{\mathbf{z}}$$

Distance between two neighboring points (*Found using the Pythagorean Theorem*):

$$ds = |d\mathbf{r}| = \sqrt{d\rho^2 + \rho^2 d\varphi^2 + dz^2}$$

Primary Curve	Primary Surface	Primary Volume
1st : Rays ⊥ to the *z-axis* (φ and z fixed, ρ varies)	Cylinder centered on the *z-axis* (ρ fixed , φ and z varies)	
2nd : Circle centered on the *z-axis* (ρ and z fixed, φ varies)	Half-plane from *z-axis* (φ fixed , ρ and z varies)	Solid Cylinder
3rd : Straight line (*z-axis*) (ρ and φ fixed, z varies)	Plane ⊥ to the *z-axis* (z fixed , ρ and φ varies)	

Primary Length Elements (dl)	Primary Area Elements (dA)	Primary Volume Elements (dV)
1st: $d\rho$ $(\hat{\boldsymbol{\rho}})$	$\rho\, d\varphi\, dz$ $(\hat{\boldsymbol{\rho}})$ (teal surface)	
2nd: $\rho\, d\varphi$ $(\hat{\boldsymbol{\varphi}})$	$d\rho\, dz$ $(\hat{\boldsymbol{\varphi}})$ (purple surface)	$\rho\, d\rho\, d\varphi\, dz$
3rd: dz $(\hat{\mathbf{z}})$	$\rho\, d\rho\, d\varphi$ $(\hat{\mathbf{z}})$ (pink surface)	

Coordinate Conversions

Cartesian → Cylindrical

$$\rho = \sqrt{x^2 + y^2}$$

$$\tan\varphi = \frac{y}{x} \quad \rightarrow \quad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z_{cyl} = z_{Cart}$$

Spherical → Cylindrical

$$\rho = r\sin\theta$$

$$\varphi_{cyl} = \varphi_{sph}$$

$$z = r\cos\theta$$

Unit Vector Conversions

Cartesian → Cylindrical

$$\hat{\mathbf{x}} = \cos\varphi\,\hat{\boldsymbol{\rho}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = \sin\varphi\,\hat{\boldsymbol{\rho}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Spherical → Cylindrical

$$\hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\rho}} - \sin\theta\,\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\varphi}}_{sph} = \hat{\boldsymbol{\varphi}}_{cyl}$$

Special Functions Involving the Del Operator (∇)

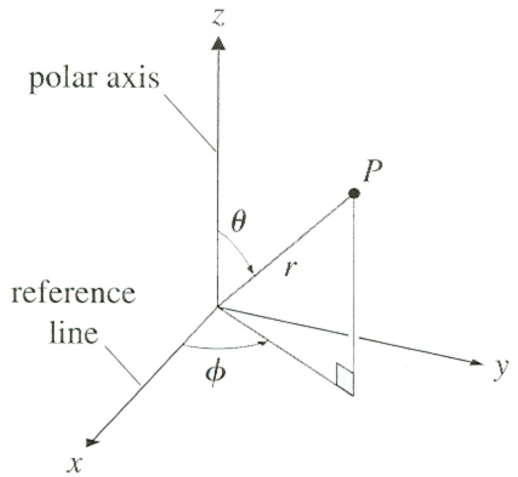
Gradient: $$\vec{\nabla} f = \hat{\boldsymbol{\rho}} \frac{\partial f}{\partial \rho} + \hat{\boldsymbol{\varphi}} \left(\frac{1}{\rho} \right) \frac{\partial f}{\partial \varphi} + \hat{\mathbf{z}} \frac{\partial f}{\partial z}$$

Divergence: $$\vec{\nabla} \cdot \mathbf{F} = \left(\frac{1}{\rho} \right) \frac{\partial}{\partial \rho} \left(\rho F_\rho \right) + \left(\frac{1}{\rho} \right) \frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}$$

Curl: $$\vec{\nabla} \times \mathbf{F} = \left[\left(\frac{1}{\rho} \right) \frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z} \right] \hat{\boldsymbol{\rho}} + \left[\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right] \hat{\boldsymbol{\varphi}} + \left(\frac{1}{\rho} \right) \left[\frac{\partial}{\partial \rho} \left(\rho F_\varphi \right) - \frac{\partial F_\rho}{\partial \varphi} \right] \hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 f = \left(\frac{1}{\rho} \right) \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \left(\frac{1}{\rho^2} \right) \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates



Variable Limits:

$$0 \le r < \infty$$

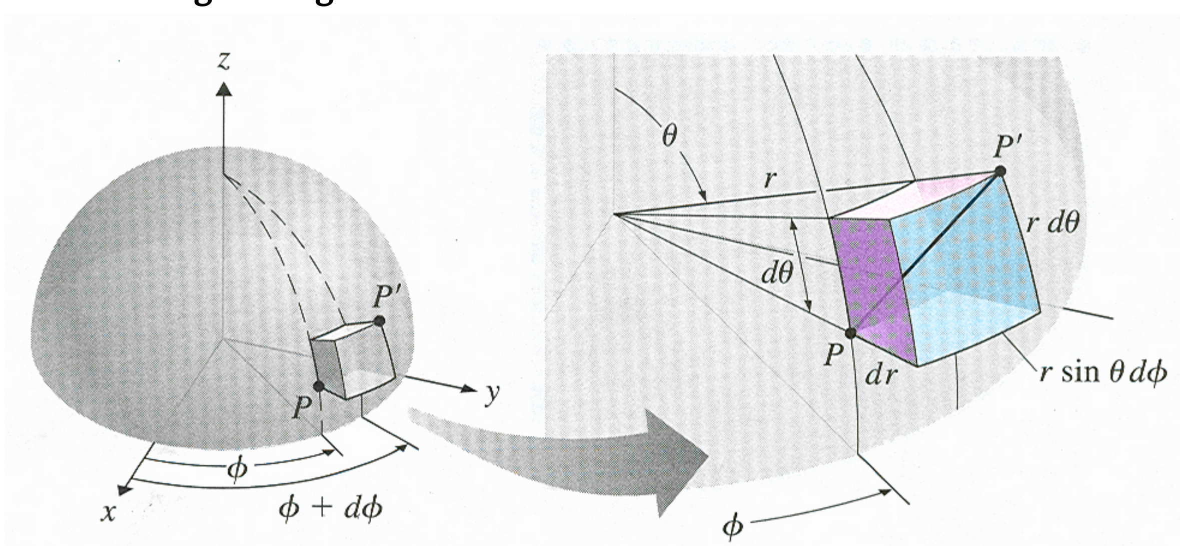
$$0 \le \theta \le \frac{\pi}{2}$$

$$0 \le \phi \le 2\pi$$

Position vector:

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad or \qquad \mathbf{r} = r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}$$

For Two Neighboring Points P and P':



Displacement between two neighboring points:

$$d\mathbf{s} = d\mathbf{r} = dr\,\hat{\mathbf{r}} + r d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta d\varphi\,\hat{\boldsymbol{\varphi}}$$

Distance between two neighboring points (*Found using the Pythagorean Theorem*):

$$ds = |d\mathbf{r}| = \sqrt{dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2}$$

Primary Element	Primary Curve	Primary Surface	Primary Volume
1st : r	Rays from the origin (θ and φ fixed, r varies)	Sphere (r fixed , θ and φ varies)	
2nd : θ	Half circle (r and φ fixed, θ varies)	Cone of half angle θ (θ fixed , r and φ varies)	Solid Sphere
3rd : φ	Circle centered on polar axis (r and θ fixed, φ varies)	Half-plane from *z*-axis (φ fixed , r and θ varies)	

Primary Length Elements (dl)		Primary Area Elements (dA)			Primary Volume Elements (dV)
1st: dr	$(\hat{\mathbf{r}})$	$r^2 \sin\theta \, d\theta \, d\varphi$	$(\hat{\mathbf{r}})$	(teal)	
2nd: $r \, d\theta$	$(\hat{\boldsymbol{\theta}})$	$r \sin\theta \, dr \, d\varphi$	$(\hat{\boldsymbol{\theta}})$	(pink)	$r^2 \sin\theta \, dr \, d\theta \, d\varphi$
3rd: $r \sin\theta \, d\varphi$	$(\hat{\boldsymbol{\varphi}})$	$r \, dr \, d\theta$	$(\hat{\boldsymbol{\varphi}})$	(purple)	

Note: The $r \sin\theta$ term is the distance from the polar axis to the projection of point P into the *xy*-plane.

Coordinate Conversions

Cartesian → Spherical

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z} \quad \to \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

or $$\cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad \to \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\tan\varphi = \frac{y}{x}$$

Cylindrical → Spherical

$$r = \sqrt{\rho^2 + z^2}$$

$$\tan\theta = \frac{\rho}{z} \quad \to \quad \theta = \tan^{-1}\left(\frac{\rho}{z}\right)$$

or $$\cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}} \quad \to \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{\rho^2 + z^2}}\right)$$

$$\varphi_{sph} = \varphi_{cyl}$$

Unit Vector Conversions

Cartesian → Spherical

$$\hat{\mathbf{x}} = \sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = \sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

Cylindrical → Spherical

$$\hat{\boldsymbol{\rho}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}$$

$$\hat{\boldsymbol{\varphi}}_{cyl} = \hat{\boldsymbol{\varphi}}_{sph}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

Special Functions Involving the Del Operator (∇)

Gradient: $$\vec{\nabla} f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\left(\frac{1}{r}\right)\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\varphi}}\left(\frac{1}{r\sin\theta}\right)\frac{\partial f}{\partial \varphi}$$

Divergence: $$\vec{\nabla}\cdot\mathbf{F} = \left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2 F_r\right) + \left(\frac{1}{r\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\, F_\theta\right) + \left(\frac{1}{r\sin\theta}\right)\frac{\partial F_\varphi}{\partial\varphi}$$

Curl:

$$\vec{\nabla}\times\mathbf{F} = \left(\frac{1}{r\sin\theta}\right)\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, F_\varphi\right) - \frac{\partial F_\theta}{\partial\varphi}\right]\hat{\mathbf{r}} + \left(\frac{1}{r}\right)\left[\left(\frac{1}{\sin\theta}\right)\frac{\partial F_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rF_\varphi\right)\right]\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\right)\left[\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}}$$

Laplacian: $$\nabla^2 f = \left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \left(\frac{1}{r^2\sin\theta}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \left(\frac{1}{r^2\sin^2\theta}\right)\frac{\partial^2 f}{\partial\varphi^2}$$