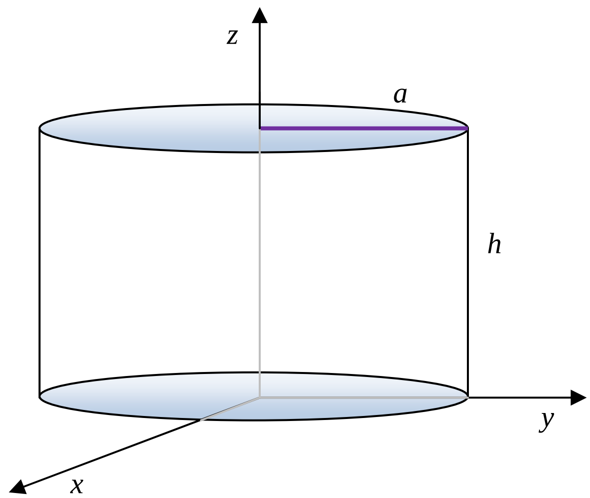
Verifying the Divergence Theorem

Let $\mathbf{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

I. Evaluate $\oint \mathbf{v} \cdot \hat{\mathbf{n}}\, dA$ over the closed surface of the following cylinder:



Need $\hat{\mathbf{n}}$:

In order to determine $\hat{\mathbf{n}}$, the problem will need to be broken down into 3 pieces (*top*, *side* & *bottom*).

TOP: $\hat{\mathbf{n}} = \hat{\mathbf{z}}$

$$\mathbf{v} \cdot \hat{\mathbf{z}} = z = h$$

$$\int_{top} h dA = h \int_{top} dA = \left(\pi a^2\right) h$$

BOTTOM: $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$

$$\mathbf{v} \cdot -\hat{\mathbf{z}} = -z = 0$$

$$\int_{bottom} 0 dA = 0$$

SIDE: $\hat{\mathbf{n}} = \dfrac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}}{a}$ (*see side box*)

$$\mathbf{v}\cdot\left(\frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}}{a}\right) = \frac{x^2 + y^2}{a} = \frac{a^2}{a} = a$$

$$\int_{side} a\,dA = a\int_{side} dA = (2\pi a h)\,a$$

To find the unit vector, Let $\phi = x^2 + y^2 - a^2$.

Since $\hat{\mathbf{n}} = \dfrac{\nabla\phi}{|\nabla\phi|}$,

➔ $$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}}}{\sqrt{(2x)^2 + (2y)^2}}$$

$$= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}}{a}$$

Combining all 3 yields:

$$\oint \mathbf{v}\cdot\hat{\mathbf{n}}\,dA = \pi a^2 h + 0 + 2\pi a^2 h = 3\pi a^2 h$$

II. Evaluate $\int_V (\nabla\cdot\mathbf{v})\,dV$ for the above cylinder.

$$\nabla\cdot\mathbf{v} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\int 3\,dV = 3\int dV = 3(\pi a^2 h)$$

➔ $$\int_V (\nabla\cdot\mathbf{v})\,dV = 3\pi a^2 h$$

∴

$$\int_V (\nabla\cdot\mathbf{v})\,dV = \oint \mathbf{v}\cdot\hat{\mathbf{n}}\,dA$$