Gaussian Integrals

For $(a > 0)$

$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{(2n-1)!!}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}} \qquad n = 0, 1, 2, 3 \ldots$$

➔ $(n = 0)$: $$\int_0^\infty e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}} \qquad n = 0, 1, 2, 3 \ldots$$

➔ $(n = 0)$: $$\int_0^\infty x e^{-ax^2} dx = \frac{1}{2a}$$

Also

$$\int_{-\infty}^\infty x^{2n} e^{-ax^2} dx = \frac{(2n-1)!!}{2^n a^n} \sqrt{\frac{\pi}{a}} \qquad n = 0, 1, 2, 3 \ldots$$

➔ $(n = 0)$: $$\int_{-\infty}^\infty e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^\infty x^{2n+1} e^{-ax^2} dx = 0$$ (*odd function over symmetric limits*)

Factorial Relations

The double factorial of a positive integer n is a generalization of $n!$ and is defined as:

$$n!! \equiv \begin{cases} n \cdot (n-2) \ldots 5 \cdot 3 \cdot 1 & n > 0 \text{ odd} \\ n \cdot (n-2) \ldots 6 \cdot 4 \cdot 2 & n > 0 \text{ even} \\ 1 & n = -1, 0. \end{cases}$$

Other useful relations:

$$\left.\begin{aligned} (2n+1)!! &= \frac{(2n+1)!}{2^n n!} \\ (2n)!! &= 2^n n! \\ (2n-1)!! &= \frac{(2n)!}{2^n n!} \end{aligned}\right\} \quad n = 0, 1, 2 \ldots$$