HWK #4

Name ________________________

Show all work. You may use Mathcad or some other software to check your answer.

1. Show that

$$\vec{\nabla} \times \left(\vec{\nabla} \Phi \right) = 0$$

for any scalar field Φ.

2. Show that

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \mathbf{F} \right) = 0$$

for any vector field **F**.

3. Calculate the divergence and the curl of each of the following vector fields:

a) $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

b) $\mathbf{V} = x^2 y\hat{\mathbf{x}} + y^2 x\hat{\mathbf{y}} + xyz\hat{\mathbf{z}}$

c) $\mathbf{V} = x \sin y\hat{\mathbf{x}} + \cos y\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$

4. Evaluate the integral

$$\oint_C \mathbf{u} \cdot d\mathbf{l}$$

by converting it to an area integral using Stoke's Theorem if:

a) C is a unit circle with ccw orientation in the *x*-*y* plane, centered at the origin and

$$\mathbf{u} = x^2 y\hat{\mathbf{x}} - xy^2\hat{\mathbf{y}}$$

b) C is a semicircle of radius *a* in the *x*-*y* plane with ccw orientation with the flat side along the *x*-axis, the center of the circle at the origin and

$$\mathbf{u} = xy^2\hat{\mathbf{x}} + x^2 y\hat{\mathbf{y}}$$

c) C is a 3-4-5 right triangle with ccw orientation with the sides of length 3 and 4 along the *x*- and *y*-axes, respectively and

$$\mathbf{u} = x^2\hat{\mathbf{x}} + xy\hat{\mathbf{y}}$$

5. Evaluate the integral

$$\int_S \mathbf{v} \cdot d\mathbf{A}$$

by converting it to a volume integral using the Divergence Theorem if:

a) S is a sphere of radius 2 centered on the origin and

$$\mathbf{v} = x^3\hat{\mathbf{x}} + 3yz^2\hat{\mathbf{y}} + 3y^2z\hat{\mathbf{z}}$$

b) S is a hemisphere of radius 1, centered at the origin, with the flat side in the *x*-*y* plane and

$$\mathbf{v} = x^2yz(\hat{\mathbf{y}} + \hat{\mathbf{z}})$$

6. Show that the vector

$$\mathbf{u} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}$$

has zero divergence and a zero curl.

7. Derive the expression for the **gradient operator** in *spherical coordinates*. Then use this result to obtain expressions for the **divergence** and **curl** in *spherical coordinates*.

Hint I: *For the divergence and curl, use an arbitrary function* $\mathbf{F}(r,\theta,\phi)$ *to help in the proof*

Hint II: *For the curl, use the relations* $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ & $\hat{\mathbf{x}}_i \dfrac{\partial}{\partial q} \times \mathbf{F} = \hat{\mathbf{x}}_i \times \dfrac{\partial \mathbf{F}}{\partial q}$