# Matrices (Part I)

Matrix – a square or rectangular array of numbers or functions that obey certain operational rules

Matrices can be denoted in 3 possible ways:

1) Using uppercase letters

A, B, C…

2) Using representative elements enclosed in brackets

$$[a_{ij}],[b_{ij}],[c_{ij}]... \qquad where \quad i=1\ldots m \quad \& \quad j=1\ldots n$$

$m$ – maximum number of rows
$n$ – maximum number of columns

3) Using a visual square or rectangular display

$$A=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad or \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

**NOTE**: Each element within the matrix has a unique label or arrangement of *i* & *j*.

## Properties of Matrices:

### Order:

The order of a matrix is indicated by listing the number of rows followed by the number of columns.

$$order = m \times n$$

### Equality:

2 matrices are equal *iff* they are the same order and the corresponding elements in each matrix are the same.

$$a_{ij}=b_{ij} \qquad where \quad i=1\ldots m \quad \& \quad j=1\ldots n$$

## Matrix Addition:

Matrices can only be added if they are the same order (*m* x *n*). When adding, you sum like components to create the new component in the resultant matrix.

$$A+B=[a_{ij}+b_{ij}]=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

*Ex.*

$$\begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}+\begin{bmatrix} -6 & 3 \\ 9 & 5 \end{bmatrix}=\begin{bmatrix} -4 & 7 \\ 8 & 5 \end{bmatrix}$$

## Zero Matrix:

A matrix whose every entry is 0, denoted O.

Properties:

$A+\mathrm{O}=A$

$A+(-A)=\mathrm{O}$

If $cA=\mathrm{O}$, then $c$ = 0 or $A$ = O

## Transpose of a Matrix:

If

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

then the transpose of A looks like

$$A^T \ or\ \tilde{A} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

and can be thought of as something like a mirror image.

**NOTE**: If A is a matrix of order $m$ x $n$, the A transpose is a matrix of order $n$ x $m$.

*Ex.* Find the transpose of the following matrices.

$$A = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 3 & 9 \\ 7 & 5 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 3 & 5 \\ 4 & 9 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 3 & -5 \end{bmatrix} \qquad C^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 4 & -5 \end{bmatrix}$$

**Properties Involving Transpose Matrices:**

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = c(A^T)$

$(AB)^T = B^T A^T$

*Ex.* Insert example(s)

## Diagonal Matrix:

A *n* x *n* (*square*) matrix in which the only non-zero elements are along the diagonal.

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

## Identity Matrix:

A diagonal matrix in which every diagonal element is 1.

$$\mathrm{I}_n \quad or \quad 1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

*n* = order of square matrix

Property:

$$1A = I_n A = A$$

## Trace:

The trace of a matrix is the sum of all the diagonal elements.

This applies only to *n* x *n* (*square*) matrices.

Properties:

$Tr(A \pm B) = Tr(A) \pm Tr(B)$

$Tr(cA) = c(Tr(A))$

$Tr(AB) = Tr(BA)$ **Note**: This is true even though $AB \neq BA$

*Ex.*

$$Tr\begin{bmatrix} 2 & 3 & 4 \\ 7 & 8 & 5 \\ 2 & 1 & 6 \end{bmatrix} = 2 + 8 + 6 = 16$$

## Scalar Multiplication:

When multiplying a matrix by a scalar, each element in the matrix gets multiplied by the scalar.

$$cA = [ca_{ij}] = \begin{bmatrix} ca_{11} & ca_{12} & \dots \\ ca_{21} & ca_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Multiplication by a scalar is commutative:

$cA = Ac$

### Matrix Subtraction

Matrix subtraction is defined as:

$$A - B = A + (-1)B = [a_{ij} - b_{ij}] = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \dots \\ a_{21} - b_{21} & a_{22} - b_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

*Ex.*

$$\begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} -6 & 3 \\ 9 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -10 & -5 \end{bmatrix}$$

$$3\begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -3 & 0 \end{bmatrix}$$

## Properties of Addition and Scalar Multiplication:

| | |
|---|---|
| $A + B = B + A$ | Commutative |
| $A + \left(B + C\right) = \left(A + B\right) + C$ | Associative |
| $(cd)A = c(dA) = d(cA)$ | |
| $1A = A$ | |
| $c\left(A + B\right) = cA + cB$ | Distributive |
| $(c + d)A = cA + dA$ | … |