# Matrices (Part II)

## Matrix Multiplication:

Matrix-Matrix multiplication (*also known as the inner product*)

$AB = C$ *iff* A is a $n$ x $m$ matrix and B is a $m$ x $p$ matrix.

The inner product is a special mathematical process that exists only if the number of columns in the first matrix is equal to the number of rows in the second matrix. The resultant matrix C is of the order $n$ x $p$.

*Ex.* Find the inner product for the following matrix:

$$\begin{bmatrix} 2 & 3 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

2 x 2 2 x 2 = 2 x 2

<u>The process for matrix-matrix multiplication is as follows</u>:

1) the $i^{th}$ elements in row of A are multiplied by the $j^{th}$ elements in column in B

2) the resultant element products are summed and represent the new $ij^{th}$ element in C

$c_{11} = 2(4) + 3(2) = 8 + 6 = 14$
$c_{12} = 2(1) + 3(-2) = 2 + -6 = -4$
$c_{21} = -1(4) + -4(2) = -4 + -8 = -12$
$c_{22} = -1(1) + -4(-2) = -1 + 8 = 7$

Therefore, the resultant matrix C looks like:

$$C = \begin{bmatrix} 14 & -4 \\ -12 & 7 \end{bmatrix}$$

*Ex.* Find the inner product for the following matrix:

$$\begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

1 x 2 2 x 2 = 1 x 2 The resultant matrix will be of order 1 *x* 2.

The form of the resultant matrix is then:

$$\begin{bmatrix} c_{11} & c_{12} \end{bmatrix}$$

$c_{11} = 3(1) + -1(2) = 3 + -2 = 1$
$c_{12} = 3(1) + -1(-2) = 3 + 2 = 5$

➔ $C = \begin{bmatrix} 1 & 5 \end{bmatrix}$

*Ex.* Find the inner product for the following matrix:

$$\begin{bmatrix} -1 & 2 & 9 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$$

2 x 3 3 x 1 = 2 x 1 The resultant matrix will be of order 2 x 1.

The form of the resultant matrix is then:

$$\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$$

$c_{11} = -1(1) + 2(-5) + 9\ (7) = -1 + -10 + 63 = 52$
$c_{21} = 0(1) + 3(-5) + 4(7) = 0 + -15 + 28 = 13$

➔ $C = \begin{bmatrix} 52 \\ 13 \end{bmatrix}$

*Ex.* Find the inner product for the previous matrices but in reverse order:

$$\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} \begin{bmatrix} -1 & 2 & 9 \\ 0 & 3 & 4 \end{bmatrix}$$

3 x 1 2 x 3 Inner dimensions not the same, there not possible!

**Properties of Matrix Multiplication:**

$(AB)C = A(BC)$

$A(B + C) = AB + AC$

$(A + B)C = AC + BC$

$c(AB) = (cA)B = A(cB)$

$AB \neq BA$ Not Commutative!!!

**NOTE**: Diagonal matrices **ARE** commutative as are a select collection of non-diagonal matrices (*which will be discussed later*).

*Ex*.

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 14 & -4 \\ -12 & 7 \end{bmatrix}$$

*But*

$$BA = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 6 & 14 \end{bmatrix}$$

**NOT EQUAL!!!**

## Inverse Matrix:

An inverse matrix is defined such that an $n$ x $n$ matrix times its inverse yields the identity matrix of order $n$ ($I_n$).

$$AA^{-1} = A^{-1}A = 1$$

Any matrix that has an inverse is called invertible (*non-singular*).

**Properties of Inverse Matrices:**

$(A^{-1})^{-1} = A$

$(A^{-1})^{T} = (A^{T})^{-1}$

$(cA)^{-1} = \frac{1}{c}A^{-1}$ $c \neq 0$

$(AB)^{-1} = B^{-1}A^{-1}$

**Finding the inverse of a Matrix**:

The traditional method is to take a matrix and adjoin it with an identity matrix and then row-reduce using Gauss-Jordan Elimination until the left side becomes the identity matrix.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc:ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{11}{8} & -\frac{7}{18} & -\frac{1}{18} \\ 0 & 1 & 0 & -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ 0 & 0 & 1 & -\frac{1}{18} & -\frac{1}{18} & \frac{5}{8} \end{array}\right]$$

Therefore

$$A^{-1} = \begin{bmatrix} \frac{11}{8} & -\frac{7}{18} & -\frac{1}{18} \\ -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{8} \end{bmatrix}$$

*Ex.* Verify that $AA^{-1} = I_3$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} \frac{11}{8} & -\frac{7}{18} & -\frac{1}{18} \\ -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{8} \end{bmatrix}$$

$$c_{11} = 3\left(\tfrac{11}{8}\right) + 2\left(\tfrac{-7}{18}\right) + 1\left(\tfrac{-1}{18}\right) = 1$$
$$c_{12} = 3\left(\tfrac{-7}{18}\right) + 2\left(\tfrac{11}{18}\right) + 1\left(\tfrac{-1}{18}\right) = 0$$
$$c_{13} = 3\left(\tfrac{-1}{18}\right) + 2\left(\tfrac{-1}{18}\right) + 1\left(\tfrac{5}{8}\right) = 0$$
$$c_{21} = 2\left(\tfrac{11}{8}\right) + 3\left(\tfrac{-7}{18}\right) + 1\left(\tfrac{-1}{18}\right) = 0$$
$$c_{22} = 2\left(\tfrac{-7}{18}\right) + 3\left(\tfrac{11}{18}\right) + 1\left(\tfrac{-1}{18}\right) = 1$$
$$c_{23} = 2\left(\tfrac{-1}{18}\right) + 3\left(\tfrac{-1}{18}\right) + 1\left(\tfrac{5}{8}\right) = 0$$
$$c_{31} = 1\left(\tfrac{11}{8}\right) + 1\left(\tfrac{-7}{18}\right) + 4\left(\tfrac{-1}{18}\right) = 0$$
$$c_{32} = 1\left(\tfrac{-7}{18}\right) + 1\left(\tfrac{11}{18}\right) + 4\left(\tfrac{-1}{18}\right) = 0$$
$$c_{33} = 1\left(\tfrac{-1}{18}\right) + 1\left(\tfrac{-1}{18}\right) + 4\left(\tfrac{5}{8}\right) = 1$$

$$\rightarrow \quad AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$