Vector Algebra

When dealing with scalars, the usual math operations (+, -, …) are sufficient to obtain any information needed. When dealing with vectors, the magnitudes can be operated on as scalars, but we must account for doing operations on or between vector directions.

Addition:

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (*commutative*)

$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (*associative*)

Subtraction:

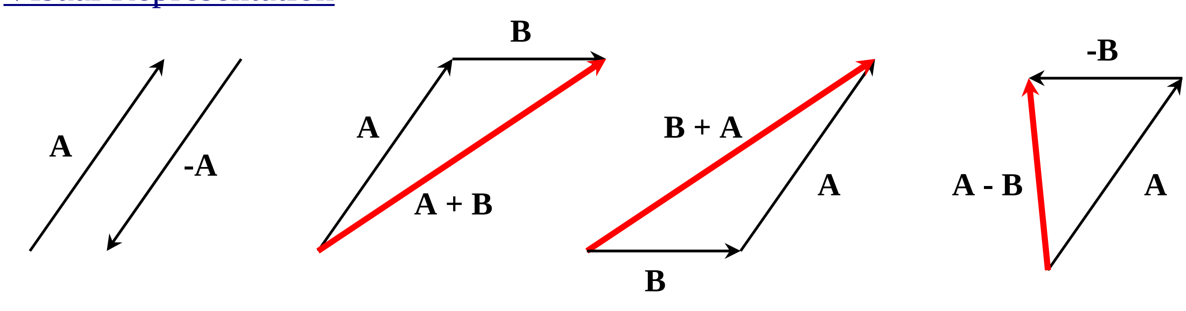
$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$

NOTE: When adding vectors, you can only combine components that are in the same direction!

Ex.

$$\langle 3,2\rangle + \langle 4,-1\rangle = (3+4)\hat{\mathbf{x}} + (2-1)\hat{\mathbf{y}}$$
$$= 7\hat{\mathbf{x}} + 1\hat{\mathbf{y}}$$

Visual Representation



< Tail to Tip Method Handout >

Magnitude or length of a vector $|\mathbf{v}|$:

Let $\mathbf{v} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}}$

The length or magnitude of $\mathbf{v}$ is the distance from the origin $\langle 0,0,0\rangle$ to point $\langle a,b,c\rangle$. Using the distance formula:

$$|\mathbf{v}| = \sqrt{(a-0)^2 + (b-0)^2 + (c-0)^2} = \sqrt{a^2 + b^2 + c^2}$$

Multiplication by a scalar:

$$a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B} \qquad (\textit{distributive})$$

If $\mathbf{A} = a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}$, then

$$k\mathbf{A} = ka_1\hat{\mathbf{x}} + ka_2\hat{\mathbf{y}} + ka_3\hat{\mathbf{z}}$$

NOTE: Multiplying a vector by a *positive* scalar does **NOT** effect its direction, only its magnitude. Multiplying by a *negative* scalar **reverses** the vector's direction.

Ex.

$$3(2\hat{\mathbf{x}} - 4\hat{\mathbf{y}} + 1\hat{\mathbf{z}}) = 6\hat{\mathbf{x}} - 12\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$$

Visual Representation

Dot Product:

I $\quad \mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3 \quad$ *where* $\mathbf{A} = a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}$ & $\mathbf{B} = b_1\hat{\mathbf{x}} + b_2\hat{\mathbf{y}} + b_3\hat{\mathbf{z}}$

II $\quad \mathbf{A} \cdot \mathbf{B} = AB\cos\theta \quad$ *where* θ *is the angle formed between the 2 vectors.*

NOTE: This process is also referred to as the ***scalar product*** because the final result is a *scalar*, not a vector.

In the second expression, *A* and *B* represent the magnitudes of the vectors **A** and **B**. This method is only used when you are looking for the angle between 2 vectors or when the angle between 2 vectors is known.

< Visual Representation of the Dot Product >

NOTE: The dot product is *commutative*, meaning $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$.
The dot product is *distributive*, meaning $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$

Ex.

$$(3\hat{\mathbf{x}} + 4\hat{\mathbf{y}}) \cdot (-1\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 5\hat{\mathbf{z}}) = (3 \cdot -1) + (4 \cdot 2) + (0 \cdot 5)$$
$$= 5$$

Ex. Given $\mathbf{A} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$ *and* $\mathbf{B} = 4\hat{\mathbf{x}} - 1\hat{\mathbf{y}}$, find the angle between.

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$

$$\theta = \cos^{-1}\left(\frac{3\cdot 4 + 2\cdot(-1)}{\sqrt{9+4}\sqrt{16+1}}\right)$$

$$\theta = \cos^{-1}\left(\frac{10}{\sqrt{221}}\right) = 0.83298 \ rad \qquad (47.7^{\circ})$$

Ex. What happens when you take the dot product of a vector with unit vectors?

Let $\mathbf{v} = v_1\hat{\mathbf{x}} + v_2\hat{\mathbf{y}} + v_3\hat{\mathbf{z}}$

$$\mathbf{v}\cdot\hat{\mathbf{x}} = \langle v_1, v_2, v_3\rangle \cdot \langle 1,0,0\rangle = v_1$$

$$\mathbf{v}\cdot\hat{\mathbf{y}} = \langle v_1, v_2, v_3\rangle \cdot \langle 0,1,0\rangle = v_2$$

$$\mathbf{v}\cdot\hat{\mathbf{z}} = \langle v_1, v_2, v_3\rangle \cdot \langle 0,0,1\rangle = v_3$$

Ex. What happens when you take the dot product between unit vectors?

$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \langle 1,0,0\rangle \cdot \langle 1,0,0\rangle = 1$ *Likewise* $\hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$

$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \langle 1,0,0\rangle \cdot \langle 0,1,0\rangle = 0$ *this means they are* $\perp$ *to each other*

likewise $\hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = 0$

NOTE: This property is the same for other coordinate systems.

* The dot product for any set of orthogonal unit vectors can be summarized by using the Kronecker delta (δ_{ij}):

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$$

Multiplication by a vector:

(Cross Product) *Let* $\mathbf{A} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3$ & $\mathbf{B} = b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2 + b_3\hat{\mathbf{e}}_3$

I $\mathbf{A}\times\mathbf{B} = (a_2b_3 - a_3b_2)\hat{\mathbf{e}}_1 - (a_1b_3 - a_3b_1)\hat{\mathbf{e}}_2 + (a_1b_2 - a_2b_1)\hat{\mathbf{e}}_3$

II $\mathbf{A}\times\mathbf{B} = \det\begin{pmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$ or $\begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

III $\mathbf{A}\times\mathbf{B} = AB\sin\theta\hat{\mathbf{n}}$ *where* $\hat{\mathbf{n}}$ *represents the direction perp. To both **A** & **B***
($\hat{\mathbf{n}}$ *= the normal unit vector, meaning perpendicular to*)

NOTE: This definition is also referred to as the ***vector product*** because the final result is a *vector*, not a scalar.

*** Important features of the cross product:**

1) It is only defined for 3-D
(for a 2-D vector, add a 0 for the missing dimension)

2) It yields a vector that is perpendicular to both original vectors
(its direction is given by the right-hand-rule)

3) The cross product obeys the following algebraic properties

$\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$ (***not** commutative*)

$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$ (*distributive*)

$c(\mathbf{A}\times\mathbf{B}) = (c\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(c\mathbf{B})$ (*distributive*)

$$\begin{cases} (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) & \text{(vector triple product)} \\ \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) & \text{(vector triple product)} \end{cases}$$

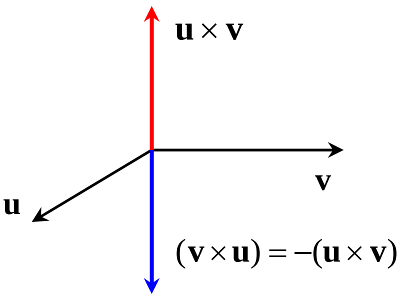
$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ (*scalar triple product*)

$\mathbf{A}\times\mathbf{A} = 0$

$\mathbf{A}\times 0 = 0$

* Geometric features of the cross product:

1) $\mathbf{u} \times \mathbf{v}$ & $\mathbf{v} \times \mathbf{u}$ are orthogonal (*perp*.) to both **u** and **v**

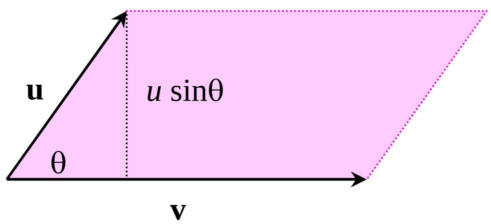


The magnitude (*length*) of **u** x **v** is a measure or reflection of how perpendicular **u** and **v** are.

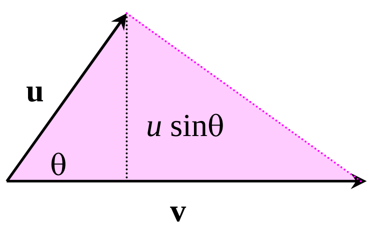
Max length is when **u** is ⊥ to **v**
0 *length* when **u** & **v** are || or anti- ||

2) $|\mathbf{u} \times \mathbf{v}| = uv\sin\theta$

3) $|\mathbf{u} \times \mathbf{v}|$ = the area of a parallelogram having **u** & **v** for adjacent sides

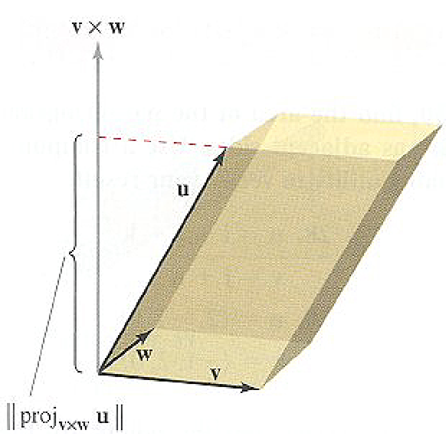


4) $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$ = the area of a triangle having **u** & **v** for adjacent sides



5) $\mathbf{u} \times \mathbf{v} = 0$ *iff* **u** & **v** are scalar multiples of each other (*parallel*)

6) $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ = the volume of a parallelepiped having **u**, **v** & **w** as adjacent edges



The triple scalar product can be found using:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

NOTE: This value could be negative!

APPLICATIONS:

The primary applications in physics involving cross products are:

Torque
Angular Momentum
Magnetic Force

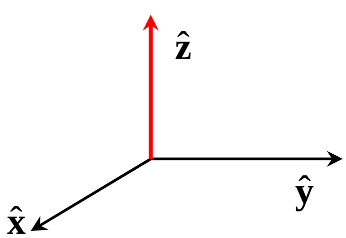
Ex. Find the cross product $\mathbf{A}\times\mathbf{B}$ for $\mathbf{A}=1\hat{\mathbf{x}}+3\hat{\mathbf{y}}+2\hat{\mathbf{z}}$ & $\mathbf{B}=3\hat{\mathbf{x}}+4\hat{\mathbf{y}}-\hat{\mathbf{z}}$

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 1 & 3 & 2\\ 3 & 4 & -1\end{vmatrix}=(-3-8)\hat{\mathbf{x}}+(6-(-1))\hat{\mathbf{y}}+(4-9)\hat{\mathbf{z}}$$

$$=\underline{-11\hat{\mathbf{x}}+7\hat{\mathbf{y}}-5\hat{\mathbf{z}}}$$

Ex. Find the cross product $\hat{\mathbf{x}}\times\hat{\mathbf{y}}$ for unit vectors $\hat{\mathbf{x}}=\langle 1,0,0\rangle$ & $\hat{\mathbf{y}}=\langle 0,1,0\rangle$

What do you expect the answer to be? $\hat{\mathbf{z}}$



$$\hat{\mathbf{x}}\times\hat{\mathbf{y}}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}=(0-0)\hat{\mathbf{x}}+(0-0)\hat{\mathbf{y}}+(1-0)\hat{\mathbf{z}}$$

$$=\hat{\mathbf{z}}$$

Ex. Find the cross product $\hat{\mathbf{y}}\times\hat{\mathbf{x}}$.

$$\hat{\mathbf{y}}\times\hat{\mathbf{x}}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & 1 & 0\\ 1 & 0 & 0\end{vmatrix}=(0-0)\hat{\mathbf{x}}+(0-0)\hat{\mathbf{y}}+(0-1)\hat{\mathbf{z}}$$

$$=-\hat{\mathbf{z}}$$

** These last 2 examples illustrate the identity $\mathbf{A}\times\mathbf{B}=-(\mathbf{B}\times\mathbf{A})$.

Ex. Find the area of a parallelogram having $\mathbf{u} = \langle 3, 2, -1 \rangle$ & $\mathbf{v} = \langle 1, 3, 3 \rangle$ as adjacent sides.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 3 & 2 & -1 \\ 1 & 3 & 3 \end{vmatrix} = (6 - (-3))\hat{\mathbf{x}} + ((-1) - 9)\hat{\mathbf{y}} + (9 - 2)\hat{\mathbf{z}}$$

$$= \underline{9\hat{\mathbf{x}} - 10\hat{\mathbf{y}} + 7\hat{\mathbf{z}}}$$

$$Area = |\mathbf{u} \times \mathbf{v}| = \sqrt{9^2 + (-10)^2 + 7^2}$$

$$= \mathbf{15.2}$$

Ex. Find the area of the parallelogram in the previous example by finding the angle between the vectors and then using $|\mathbf{u} \times \mathbf{v}| = uv \sin\theta$.

To find the angle, we use the dot product:

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{uv}\right)$$

$$\theta = \cos^{-1}\left(\frac{3 \cdot 1 + 2 \cdot 3 + (-1) \cdot 3}{\sqrt{9 + 4 + 1}\sqrt{1 + 9 + 9}}\right)$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{266}}\right) = 68.4^o \quad or \quad (.38\pi)$$

Therefore:

$$Area = \sqrt{266}\sin(68.4^o)$$

$$= \mathbf{15.2}$$

Applications of Dot and Cross-products

Work

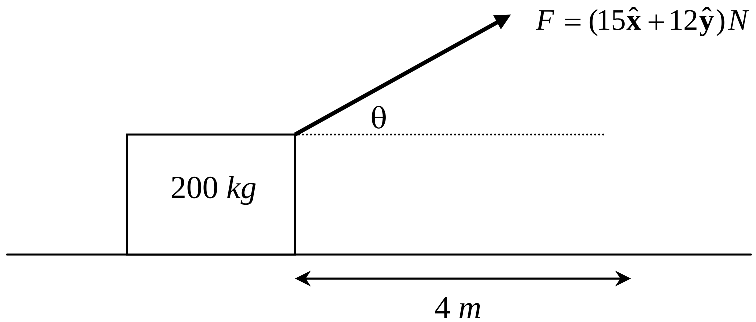
Work done by a constant force is defined as $W = \mathbf{F} \cdot \mathbf{d} = Fd\cos\theta$.

In other words, how much work is done on an object is equal to the magnitude of the applied force in the direction of motion.

The work done on an object is also a measure of the amount of energy the object has *gained* (W > 0) or *lost* (W < 0).

Ex.

How much work is required to move a 200 *kg* crate 4 *m* if it is being dragged by a steel cord under a force $F = (15\hat{\mathbf{x}} + 12\hat{\mathbf{y}})N$?



Method I: (direct dot product)

$$W = \mathbf{F} \cdot \mathbf{d}$$

$$= \left(15\hat{\mathbf{x}} + 12\hat{\mathbf{y}}\right) \cdot 4\hat{\mathbf{x}}$$

$$= \left(15\hat{\mathbf{x}} \cdot 4\hat{\mathbf{x}}\right) + \left(12\hat{\mathbf{y}} \cdot 4\hat{\mathbf{x}}\right)$$

$$= 60\left(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}}\right) + 48\left(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\right)$$

NOTE: $\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1$ & $\hat{\mathbf{y}} \cdot \hat{\mathbf{x}} = 0$

= 60 J

Method II: (using magnitudes and angles)

$$W = Fd\cos\theta$$

$$= \left(\sqrt{369}\right)\left(4\right)\cos\left(38.66^{o}\right)$$

= 60 J

$$F = \sqrt{15^2 + 12^2} = \sqrt{369}$$

$$d = 4$$

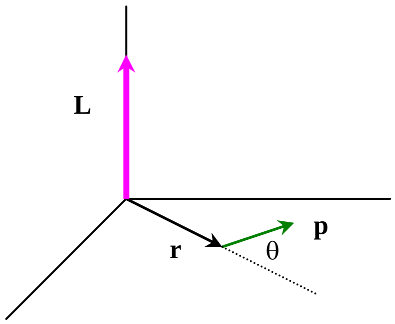
$$\theta = \tan^{-1}\left(\frac{12}{15}\right) = 38.66^{o}$$

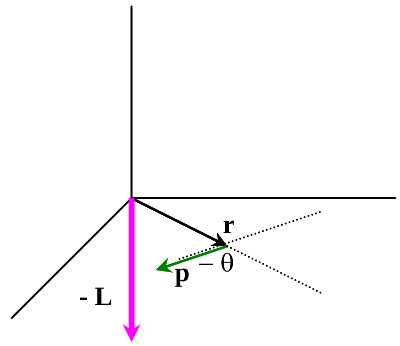
Angular Momentum

The angular momentum of a rotating object about a fixed point is given by $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$, where **r** is the displacement from the fixed point to a point on the object and **p** is the linear momentum of the point located at **r** (or m is the mass and **v** is the velocity of a point located at **r**).

The physical interpretation of angular momentum is two-fold:

1) **L** is an indicator of the direction an object is rotating.
   L > 0 *Counter-Clockwise*
   L < 0 *Clockwise*





2) |**L**| is a measure of how hard it is to stop an object that is rotating.

Special Case: Circular Motion

Since the angular momentum is a cross-product, the magnitude is given by:

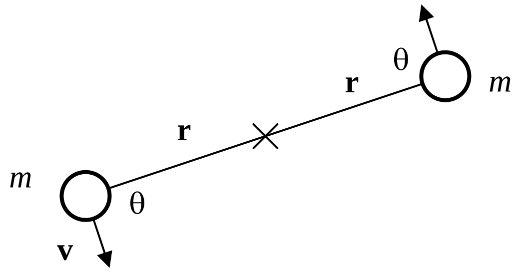
$$L = rp\sin\theta = mvr\sin\theta$$

Since $\sin\theta$ can range from -1 to 1, this means the value of L can range from $-mvr$ to mvr.

➔ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad -1 \le \sin\theta \le 1$ $\quad -mvr \le L \le mvr$

The maximum values (1 *or* -1) occur when $\theta = 90^{o}$ or -90^{o}. These special angles refer to an object experiencing circular motion.

Ex. Diatomic Molecule



$\theta = 90^o$

Find $|\mathbf{L}|$

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$$

$$|\mathbf{L}| = |\mathbf{L}_1 + \mathbf{L}_2|$$

$$L = mvr\sin\theta + mvr\sin\theta$$

$$L = mvr(1) + mvr(1)$$

$$L = 2mvr$$