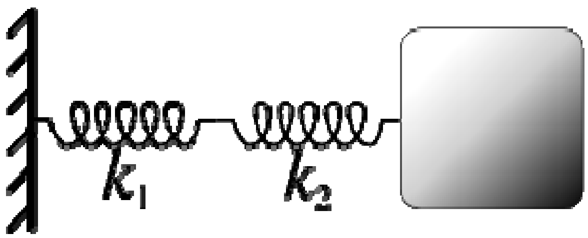
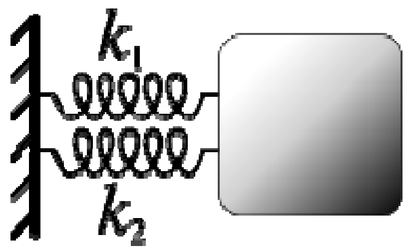
2 Springs in Series and Parallel

The following table compares the values of two spring configurations attached to a mass and compressed.

Comparison	In Series	In Parallel
		
Equivalent spring constant	$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$	$k_{eq} = k_1 + k_2$
Compressed distance	$\frac{x_1}{x_2} = \frac{k_2}{k_1}$	$x_1 = x_2$
Energy stored	$\frac{E_1}{E_2} = \frac{k_2}{k_1}$	$\frac{E_1}{E_2} = \frac{k_1}{k_2}$

Equivalent Spring Constant (*Series*)

Deriving k_{eq} in the series case is a little trickier than in the parallel case. Defining the equilibrium position of the block to be x_2, we'll be looking for equation for the force on the block that looks like:

$$F_b = -k_{eq} x_2.$$

To begin, we'll also define the equilibrium position of the point between the two springs to be x_1.

The force on the block is

$$F_b = -k_2 (x_2 - x_1) . \qquad (1)$$

Meanwhile, the force on the point between the two springs is

$$F_s = -k_1 x_1 + k_2(x_2 - x_1).$$

Now, when the block is pushed so the springs are compressed and the system is allowed to come to equilibrium, the force between the springs must sum to zero, so with $F_s = 0$ we can solve for x_1:

$$\begin{aligned}
&-k_1 x_1 + k_2(x_2 - x_1) = 0 \\
&-k_1 x_1 - k_2 x_1 = -k_2 x_2 \\
&(k_1 + k_2)\, x_1 = k_2 x_2
\end{aligned}$$

so

$$x_1 = \frac{k_2}{k_1 + k_2} x_2.$$

Now we just plug this back into (1):

$$\begin{aligned}
F_b &= -k_2 x_2 + k_2 x_1 \\
&= -k_2 x_2 + k_2 \left(\frac{k_2}{k_1 + k_2} x_2 \right) \\
&= -k_2 x_2 \left(\frac{k_1 + k_2}{k_1 + k_2} \right) + \frac{k_2^2}{k_1 + k_2} x_2 \\
&= x_2 \frac{-k_1 k_2 - k_2^2 + k_2^2}{k_1 + k_2}
\end{aligned}$$

Finally, the force on the block has been found:

$$F_b = -\left(\frac{k_1 k_2}{k_1 + k_2} \right) x_2.$$

So we can define everything in the parenthesis to be

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}.$$

Which can also be written:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

Equivalent Spring Constant (*Parallel*)

Both springs are touching the block in this case, and whatever distance spring 1 is compressed has to be the same amount spring 2 is compressed.

The force on the block is then:

$$\begin{aligned} F_b &= F_1 + F_2 \\ &= -k_1 x - k_2 x \end{aligned}$$

So the force on the block is

$$F_b = -(k_1 + k_2)x.$$

Thus, we can define the equivalent spring constant as

$$k_{eq} = k_1 + k_2\,.$$

Compressed Distance (*Series*)

In the case of the parallel springs, the compression distance is obviously the same. The situation is slightly more complex for springs in series. When two springs are in series, the magnitude of the force of the springs on each other is equal:

$$\begin{aligned} &|F_1| = |F_2| \\ &k_1 x_1 = k_2 \left(x_2 - x_1\right). \end{aligned}$$

For spring 1, x_1 is the distance from equilibrium length, and for spring 2, x_2 - x_1 is the distance from its equilibrium length. So we can define

$$\begin{aligned} a_1 &= x_1 \\ a_2 &= x_2 - x_1. \end{aligned}$$

Plug these definitions into the force equation, and we'll get a relationship between the compressed distances for the **in series** case:

$$\frac{a_1}{a_2} = \frac{k_2}{k_1}.$$

<u>Energy Stored</u> (*Series*)

* For the **series** case, the ratio of energy stored in springs is:

$$\frac{E_1}{E_2} = \frac{\frac{1}{2}k_1 a_1^2}{\frac{1}{2}k_2 a_2^2},$$

but a there is a relationship between a_1 and a_2 derived earlier, so we can plug that in:

$$\frac{E_1}{E_2} = \frac{k_1}{k_2}\left(\frac{k_2}{k_1}\right)^2 = \frac{k_2}{k_1}.$$

* For the **parallel** case,

$$\frac{E_1}{E_2} = \frac{\frac{1}{2}k_1 x^2}{\frac{1}{2}k_2 x^2}$$

because the compressed distance of the springs is the same, this simplifies to

$$\frac{E_1}{E_2} = \frac{k_1}{k_2}.$$