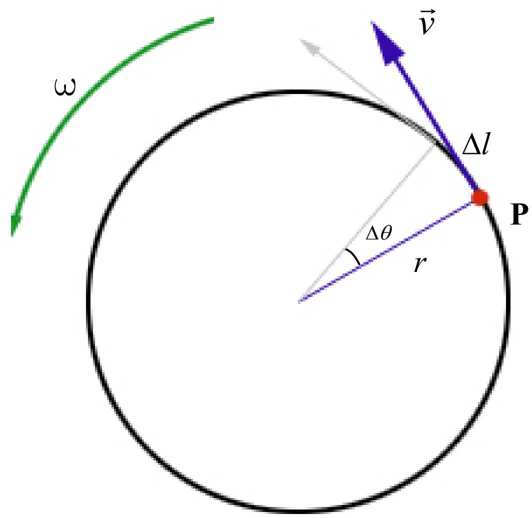
Relating Angular Motion and Linear Motion Models

Consider:



Linear/Angular Speed:

For a given point P, there is a tangential linear velocity ($\vec{v}$) at time t. At some time Δt later, the point has rotated through some angle $\Delta\theta$. If $\Delta\theta$ is small, we can approximate the arc length Δs with a linear displacement Δl.

Using the definitions of speed and angular speed:

$$v = \frac{\Delta l}{\Delta t} \qquad \& \qquad \omega = \frac{\Delta \theta}{\Delta t}$$

and equating through Δt, we get:

$$\frac{\Delta l}{v} = \frac{\Delta \theta}{\omega}$$

Solving for v we get:

$$v = \omega \frac{\Delta l}{\Delta \theta}$$

In the limit as $\Delta\theta \rightarrow 0$, $\Delta\theta \rightarrow d\theta$ and $\Delta l \rightarrow dl$

$$v = \omega \frac{dl}{d\theta}$$

From $\theta = \frac{s}{r}$ *and noting that r is constant:*

$$d\theta = \frac{1}{r}ds \quad \rightarrow \quad r = \frac{ds}{d\theta}$$

In our approximation, ds = dl

$$\rightarrow \quad r = \frac{dl}{d\theta}$$

&

$$v = \omega r \quad or \quad \omega = \frac{v}{r}$$

Linear/Angular acceleration:

Using $a = \frac{dv}{dt}$ *and substituting in* $v = \omega r$

$$a = \frac{d(\omega r)}{dt} = r\frac{d\omega}{dt} \qquad but \qquad \alpha = \frac{d\omega}{dt}$$

$$\rightarrow \quad a = r\alpha$$

Centripetal acceleration:

Using $a_r = \frac{v^2}{r}$ *and substituting in* $v = \omega r$

$$a_r = \frac{(\omega r)^2}{r}$$

$$\rightarrow \quad a_r = r\omega^2$$