Modeling Non-Uniform Motion

Windmills, which are used in the great plains of Holland and North Germany to supply the want of falling water, afford another instance of the action of velocity. The sails are driven by air in motion - by wind.

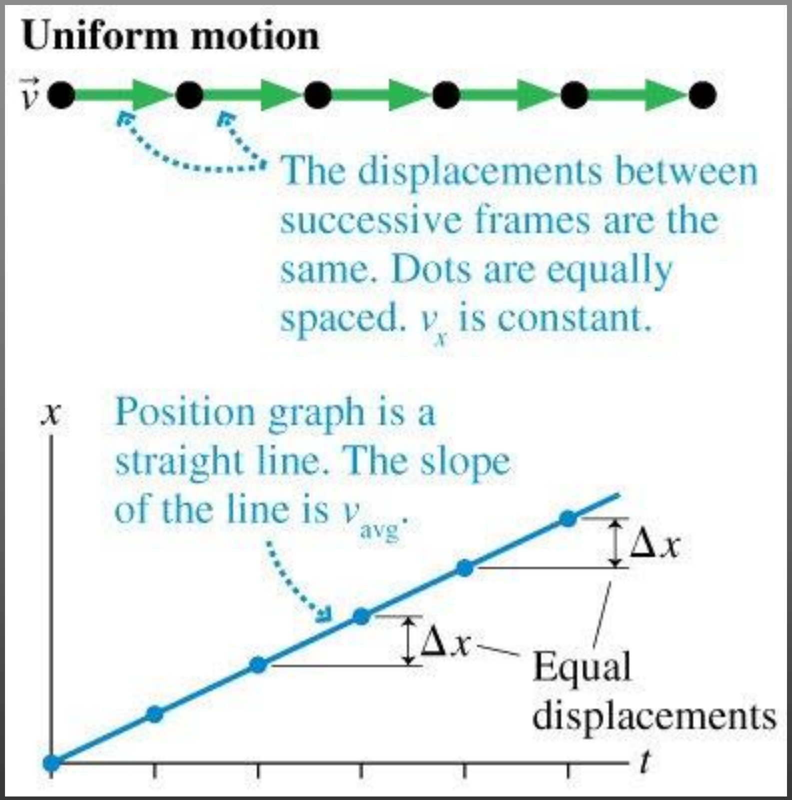
-- Hermann von Helmholtz

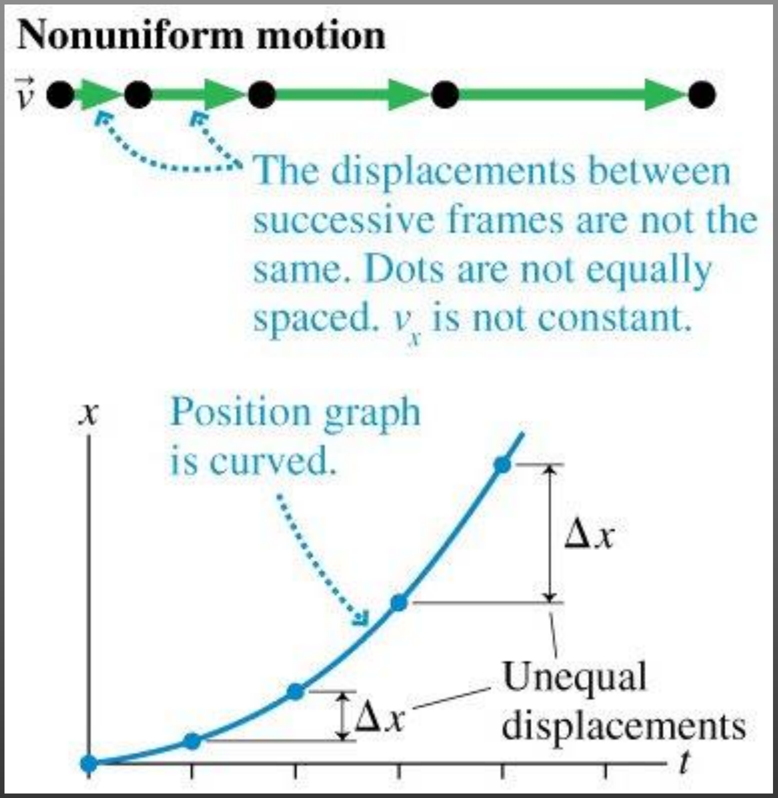
Non-Uniform Motion

- Very few objects in the universe move with constant velocity (uniform motion).

- In non-uniform motion, the change in position during equal time intervals is different (increases or decreases).

Comparing Uniform & Non-Uniform Motion





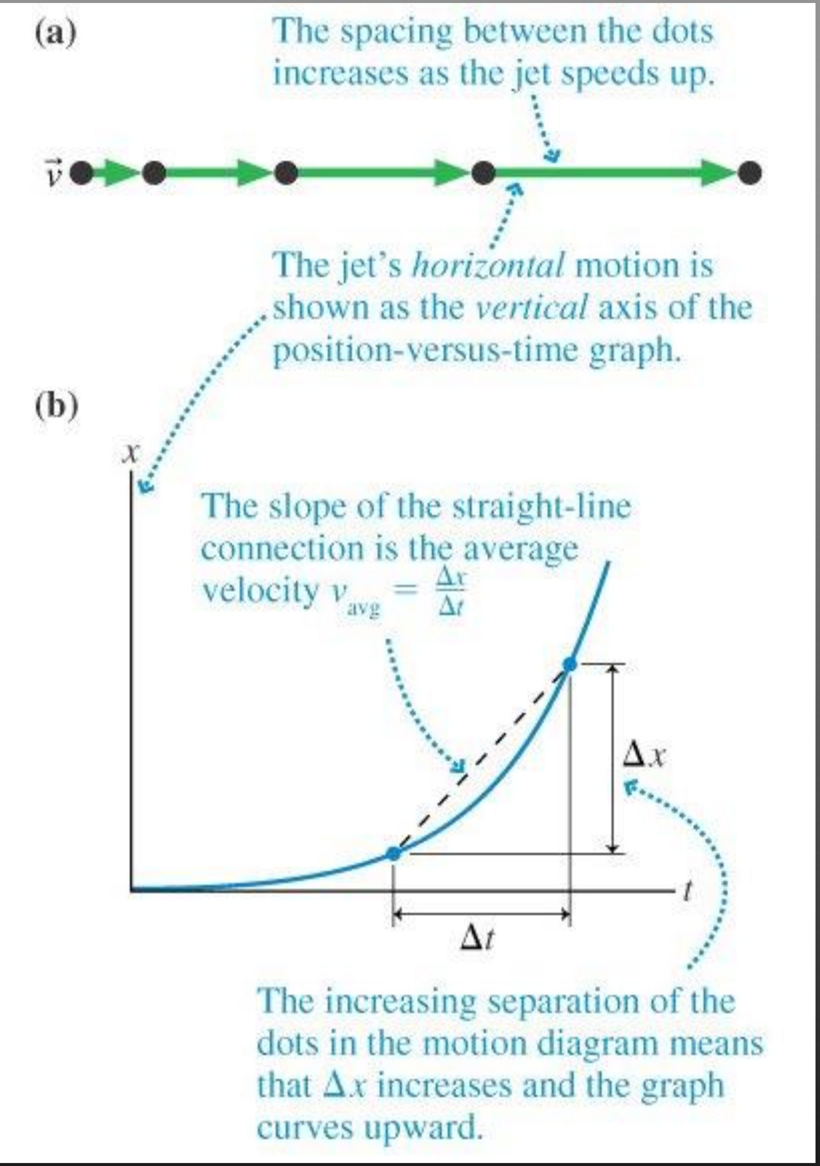
- In Uniform motion, the velocity is the same at all times (constant).
- In Non-Uniform motion, the velocity of the object is constantly changing.
- Is it possible to determine the velocity at **ANY** point in time for an object undergoing non-uniform motion?

YES!

Instantaneous Velocity

- The magnitude of the velocity vector at a specific point in time is called the **instantaneous velocity**.

- Consider a jet plane taking off from the deck of an aircraft carrier.



The motion diagram in part (*a*) clearly shows that the **velocity** of the plane is increasing during each successive time interval.

The position graph in part (*b*) shows how the change in **position** is increasing in time.

The average velocity in 1 D between two positions is:

$$v_x \equiv \frac{\Delta x}{\Delta t}$$

(*the slope of the line that passes through **both** points*)

Example

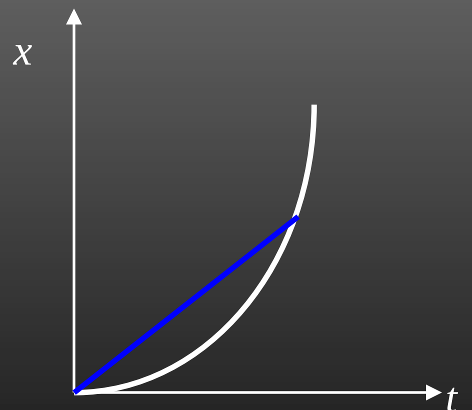
Find the ***average velocity*** (v_x) between 0 and 3 seconds for an object experiencing non-uniform motion if the position function is given by:

$$x(t) = t^2$$

Motion Analysis:

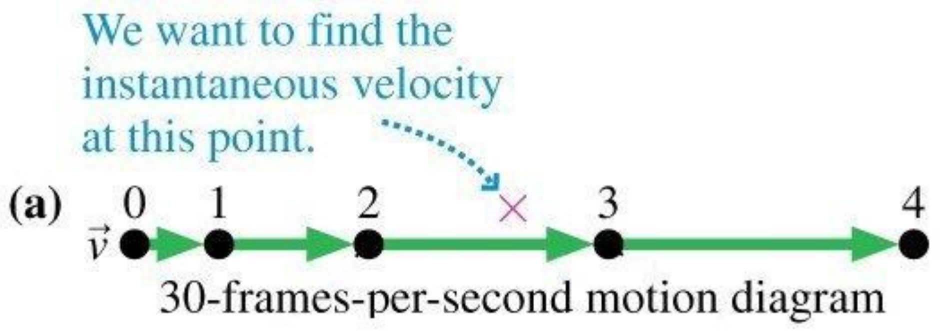
Since $x(t)$ is positive at all times, the object is always moving to the right.

Position Graph



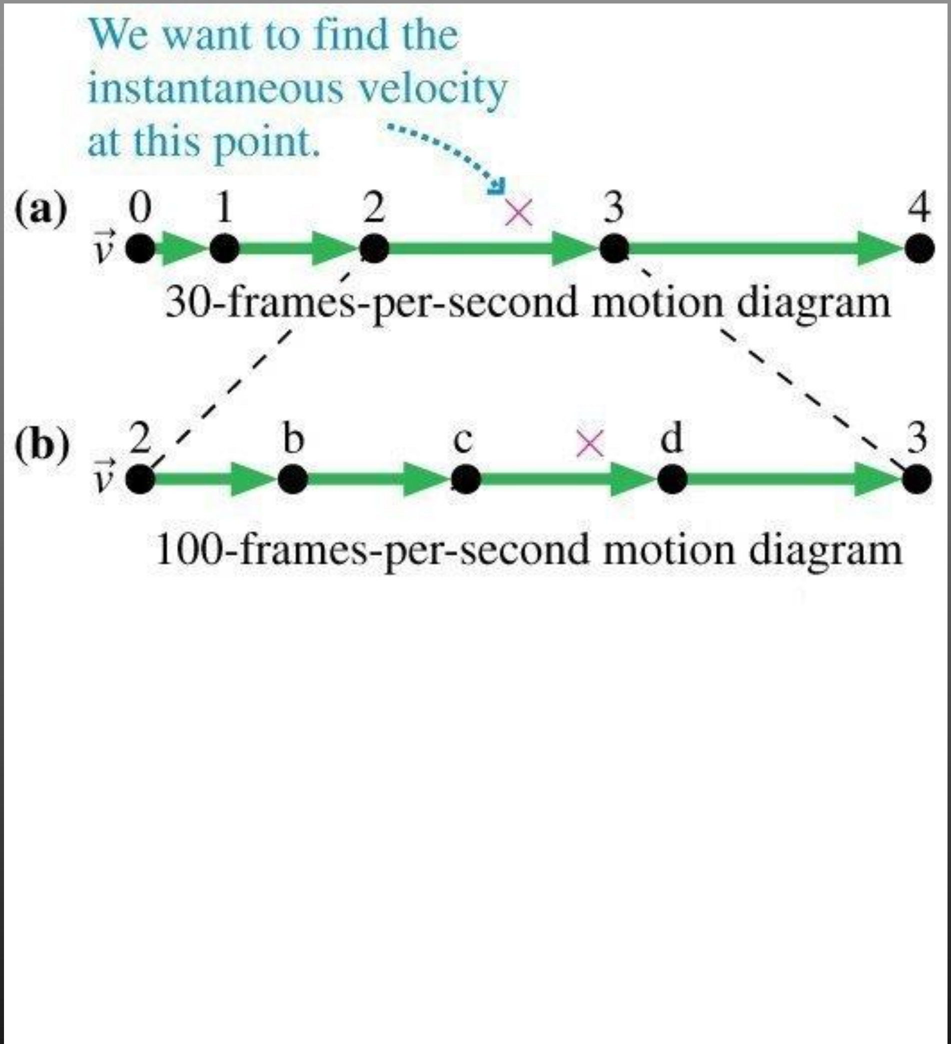
$$\begin{aligned}(v_x)_{avg} &= \frac{\Delta x}{\Delta t} \\ &= \frac{x(t_f) - x(t_i)}{t_f - t_i} \\ &= \frac{x(3) - x(0)}{3 - 0} \\ &= 3\,\frac{m}{s}\end{aligned}$$

Analyze the velocity of the plane at the point marked with an X.



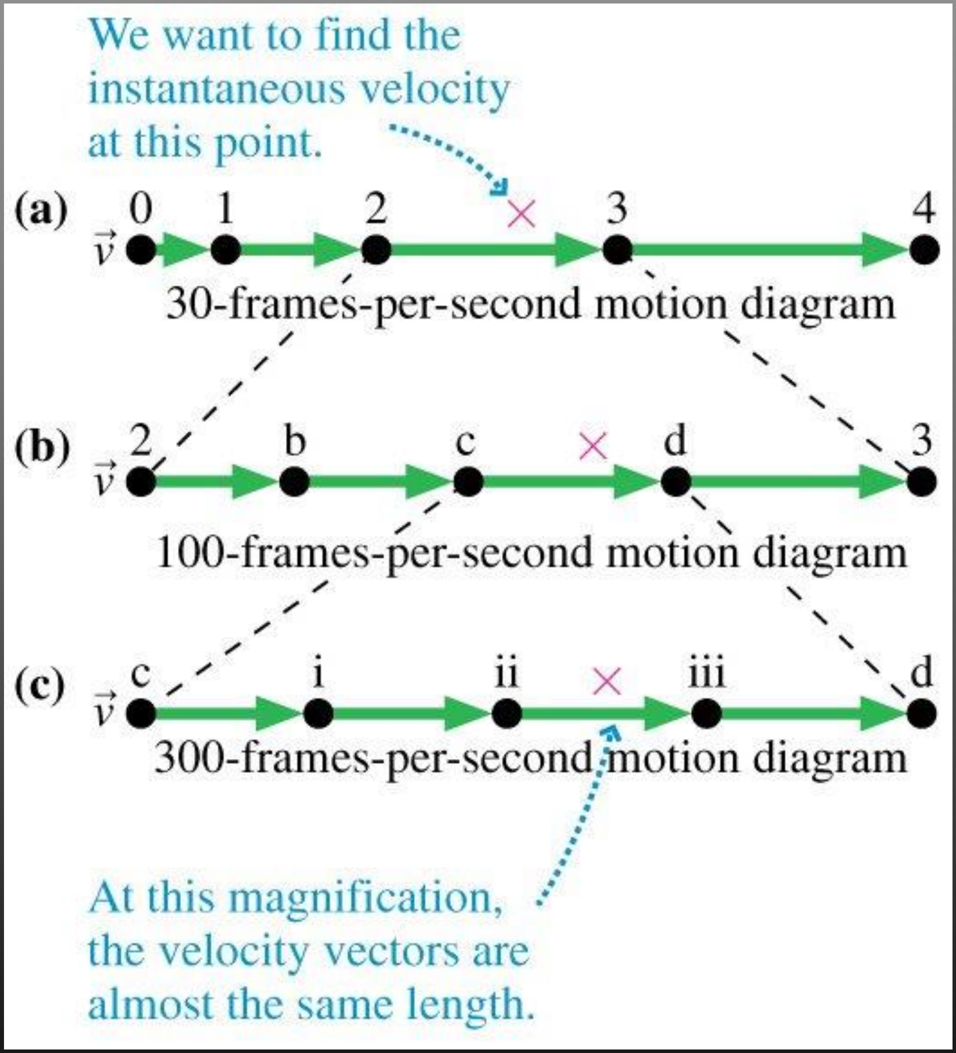
(a) A motion diagram made using a normal 30 fps camera.

The green velocity vector between points 2 & 3 represents the **average velocity** during that time interval.



(b) A Motion diagram made using a 100 fps camera filming just the segment between points 2 & 3.

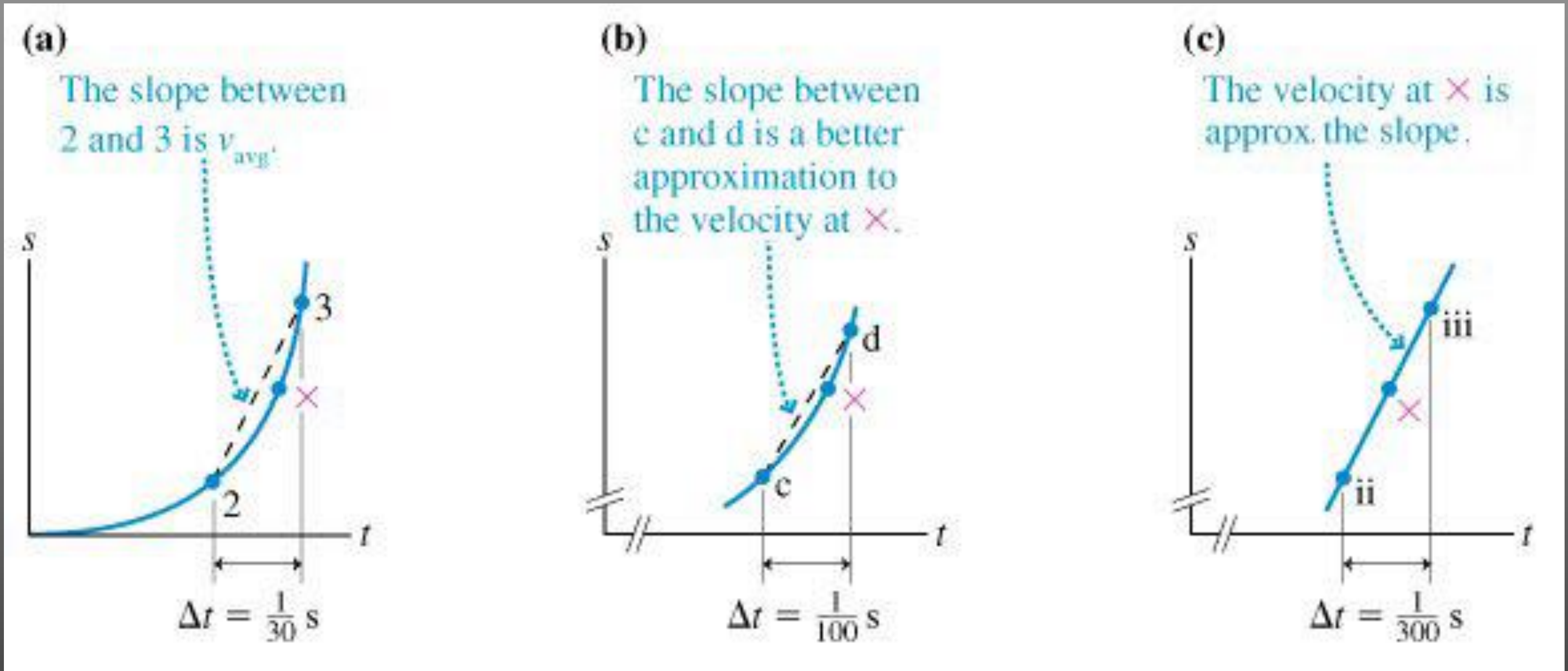
This "zoomed in" view shows another set of average velocities between points 2 & 3, but these are a little closer to being uniform in length.



(c) A motion diagram made using a 300 fps camera filming just the segment between points *ii* & *iii*.

At this magnification, all of the average velocity arrows are ***nearly*** identical. Thus, the instantaneous velocity would have the same value as any one of the average velocity vectors within this interval!

Visually, our analysis looks like this:



The only way to ever have the correct value for the instantaneous velocity is for Δt to become very, very small. In mathematical terms, $\Delta t \rightarrow 0$.

Mathematical Definition of Instantaneous Velocity

- Our analysis yields:

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \text{instantaneous velocity}$$

- But from Calculus, we know that:

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\boldsymbol{dx}}{\boldsymbol{dt}}$$

- Thus, the model for calculating instantaneous velocity in 1-D is:

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\boldsymbol{dx}}{\boldsymbol{dt}} \equiv \text{instantaneous velocity}$$

Example

- The position of a 1-D particle is given by the following position function:

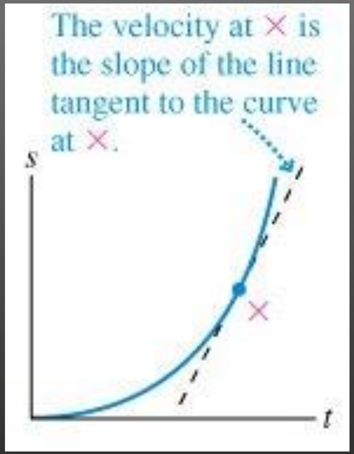
$$x(t) = \left(-t^3 + 3t\right) m$$

  - What is the particles position at $t = 2\ s$?
    - -2 m
  - What is the particles velocity as a function of time?
    - $(-3t^2 + 3)$ m/s
  - What is the particles velocity at t = 2 s?
    - -9 m/s

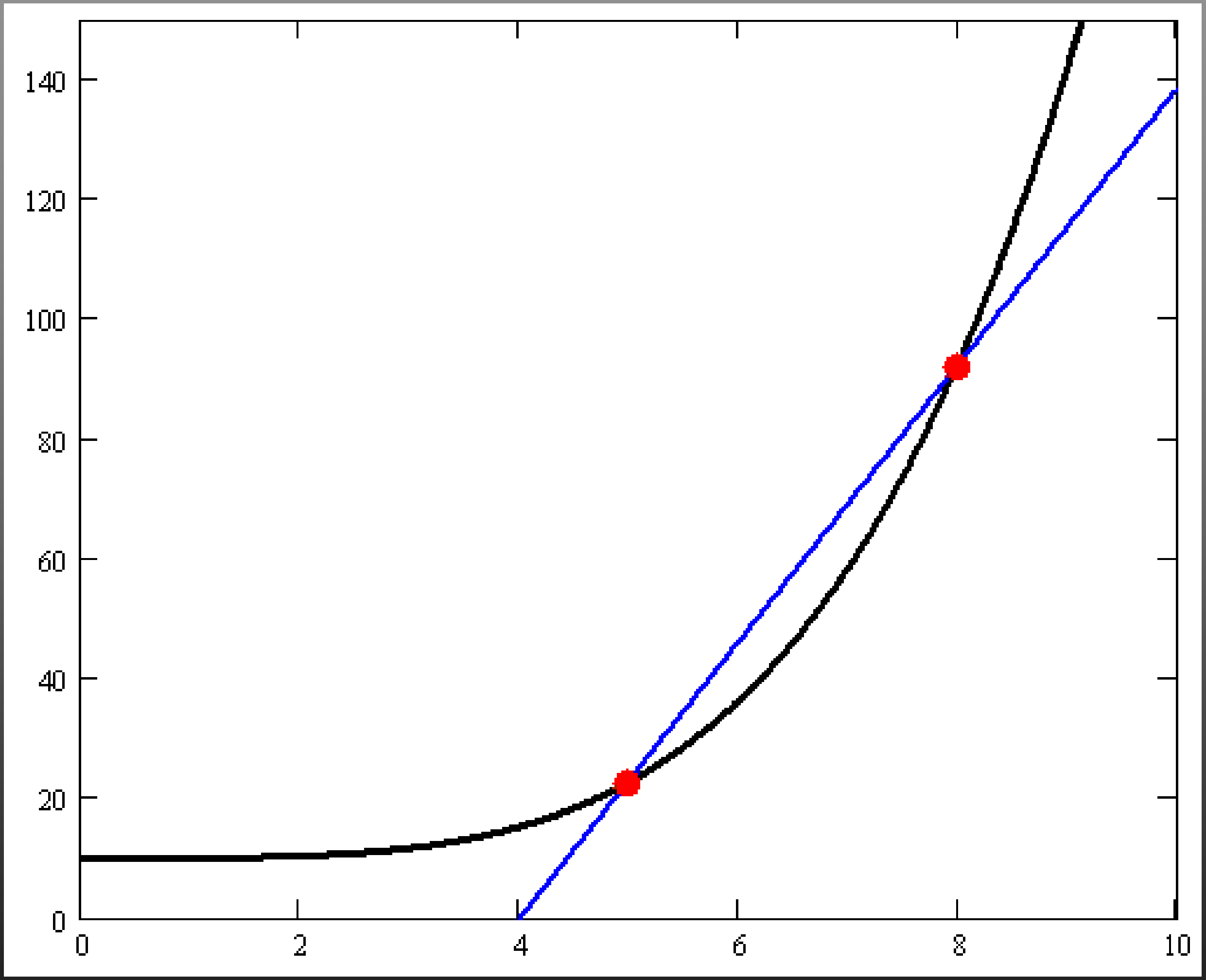
Graphical Definition of Instantaneous Velocity

- As Δt gets smaller, the slope of the line between two points on a position graph approaches the slope of a line at a **SINGLE** point.

The instantaneous velocity at time t is the slope of the line that is tangent to a position *vs.* time curve on a position graph **AT** time t.



Video of the limiting process.



Pitfall Prevention

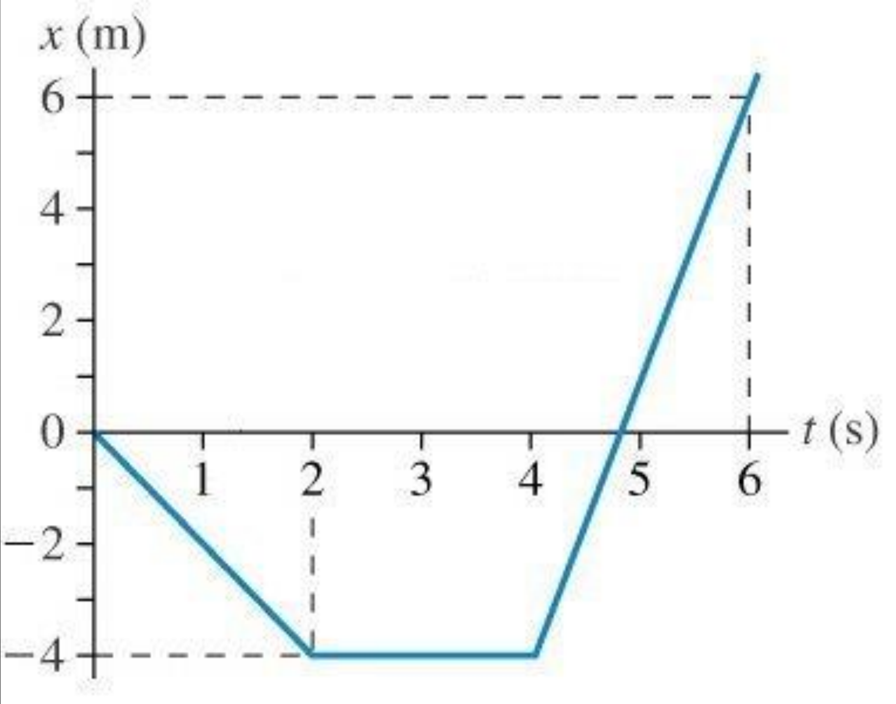
- **Average speed ≠ average velocity** in general, but the magnitude of the instantaneous speed **ALWAYS** equals the magnitude of the instantaneous velocity!

Reason: Distance = Displacement at infinitesimal scales

Example

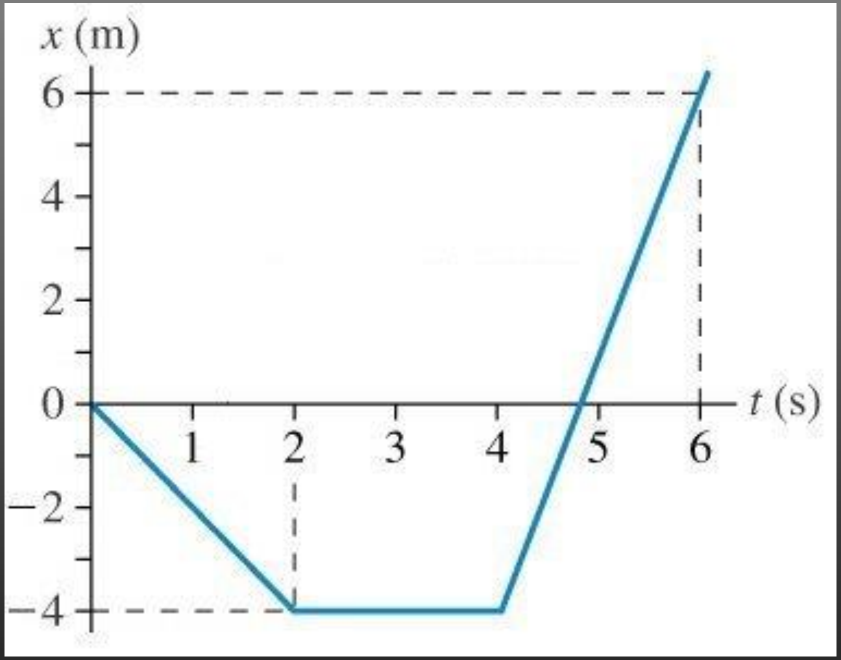
The following is a position graph of a car:

- Determine the car's velocity vs. time graph
- Describe the motion of the car



Determining the car's velocity vs. time graph

- Since the car's position graph is a series of straight lines, the car is moving in periods of uniform motion.



Slope from 0 s to 2 s:

-2.0 m/s

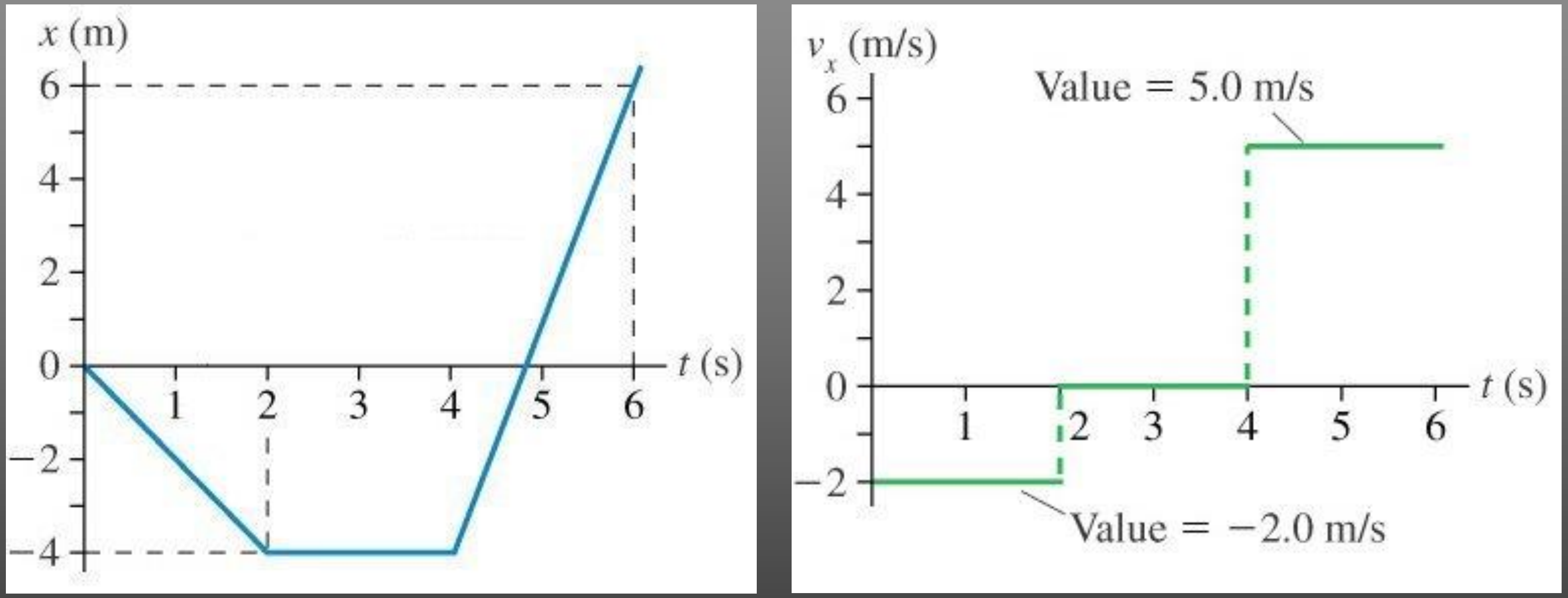
Slope from 2 s to 4 s:

0.0 m/s

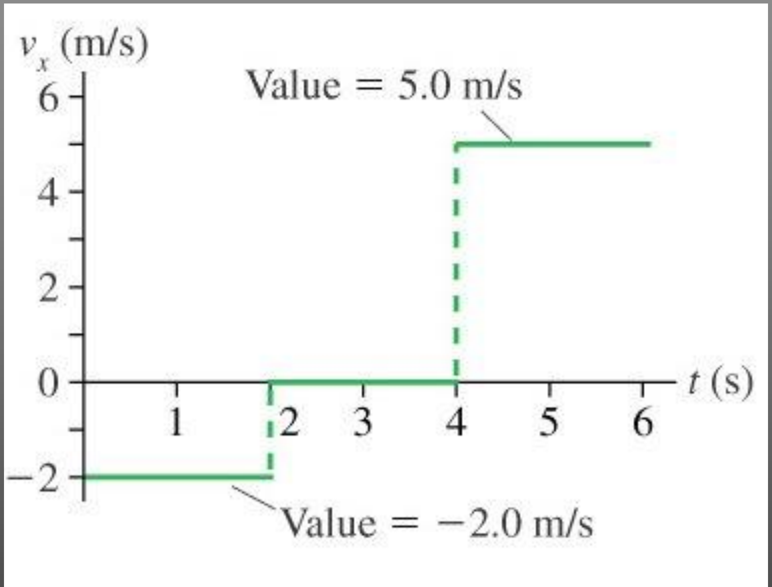
Slope from 0 s to 6 s:

5.0 m/s

- Graphing these velocity values for their corresponding time periods yield:



Describing the motion of the car



From 0 s to 2 s

- $v < 0$, the car is **backing up** at a constant velocity

From 2 s to 4 s

- $v = 0$, the car is at **rest** or **stationary**

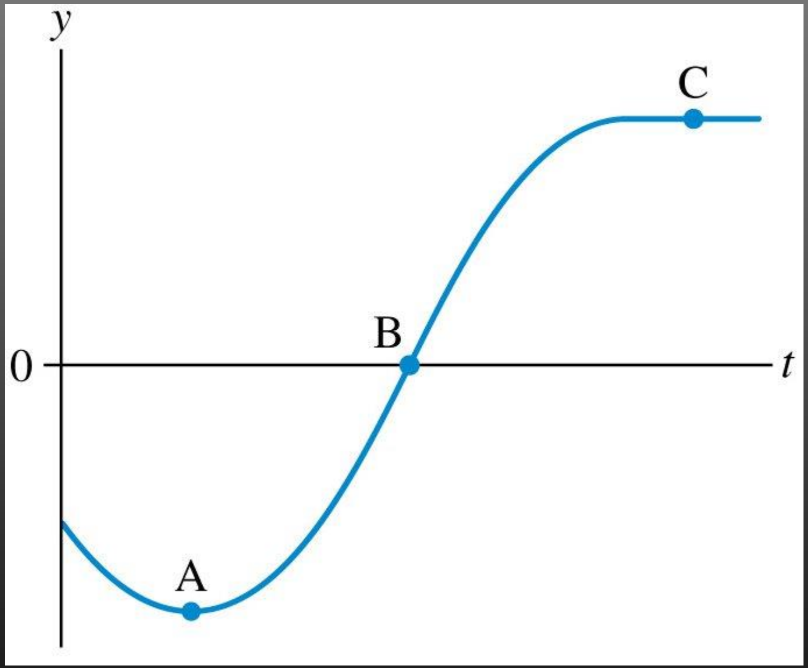
From 4 s to 6 s

- $v > 0$, the car is **moving forward** at a constant velocity

Example

The following is a position graph of a particle:

- At which labeled point(s) is the particle moving the slowest?
- At which labeled point(s) is the particle moving the fastest?



Slowest: A & C

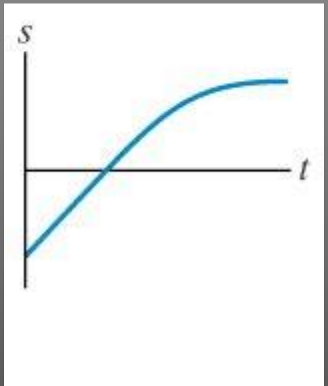
The magnitude of the slope at these points is zero, which means the **velocity** is zero.

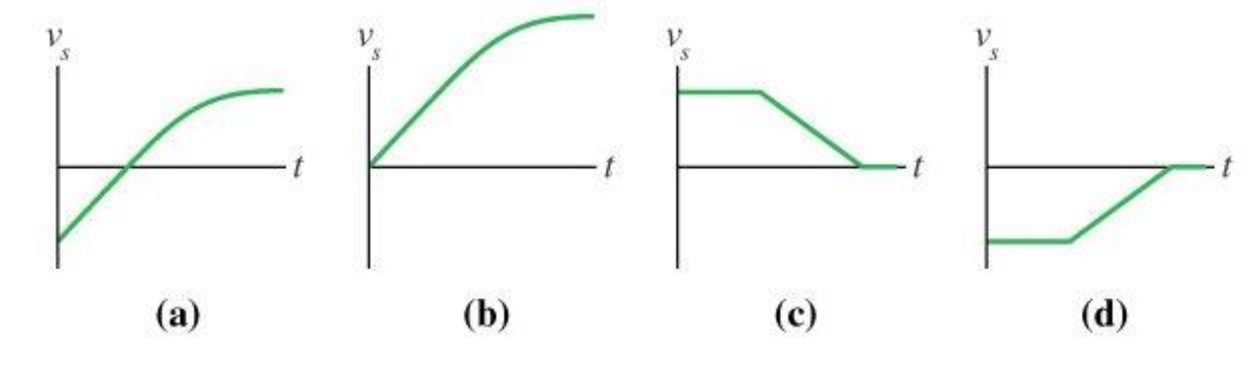
Fastest: B

The magnitude of the slope is max at this point.

Knowledge Inventory

- Which velocity vs. time graph best relates to the following position graph?





(c)

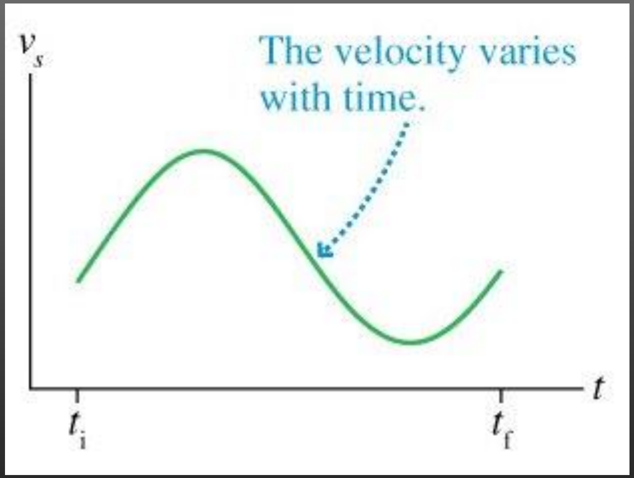
Finding Position From Velocity

Suppose we are given a velocity function or velocity vs. time graph, how could we determine the position of an object?

Example:

$$v(t) = 3t$$

or



Position from a Velocity Function

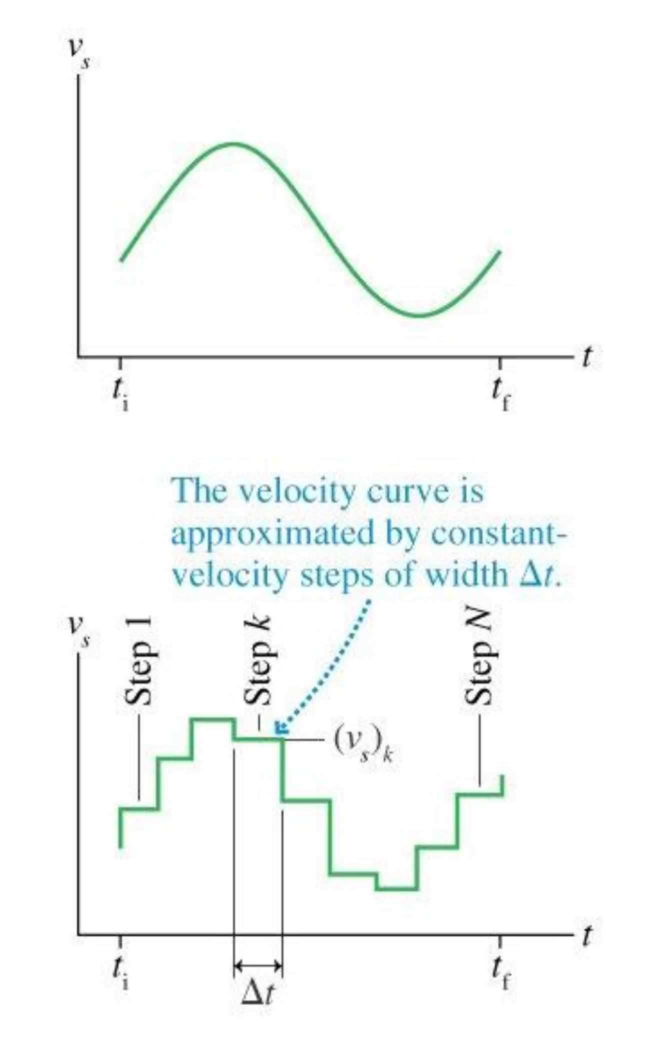
v(*t*) = velocity at any instant *t*

➔ $$\frac{dx}{dt} = v(t)$$

Rearranging and using some calculus …

$$x(t) = \int v(t)\, dt$$

Position from a Velocity Graph



- Using this approximation:

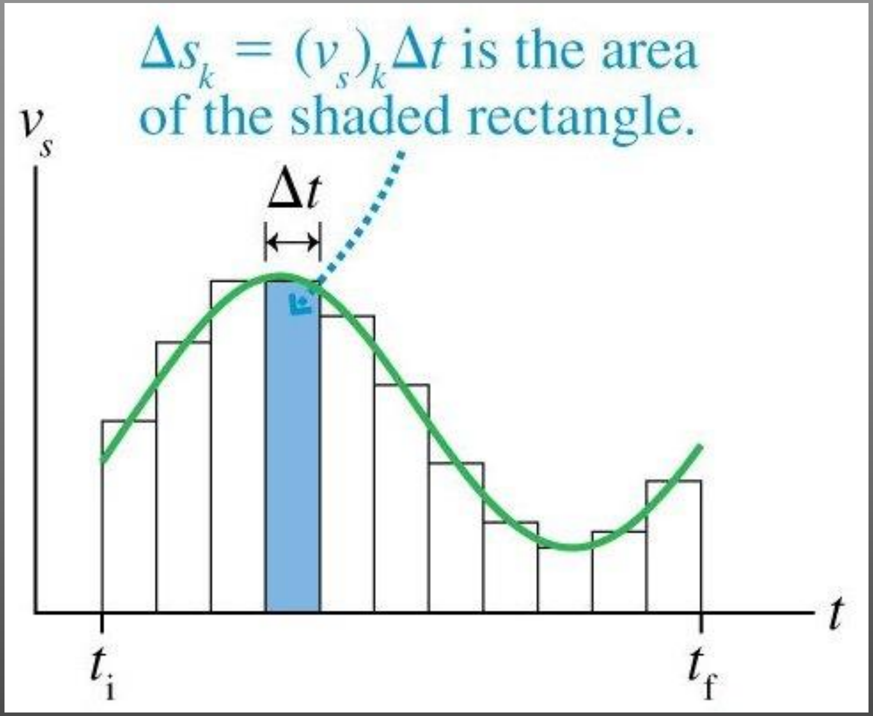
  The displacement during step *k* would be:

$$\Delta s_k = (v_s)_k \Delta t$$

  The total displacement would be:

$$\Delta s_k = \sum_{k=1}^{N} (v_s)_k \Delta t$$

Visually,



- This approximation for the total displacement gets better the smaller you make Δt.

In General…

- The correct value of the displacement occurs when $\Delta t \rightarrow 0$ and the number of steps N → infinity.

$$\Delta s = \lim_{\Delta t \to 0} \sum_{k=1}^{N} (v_s)_k \Delta t = \int_{t_o}^{t_f} v_s(t)dt$$

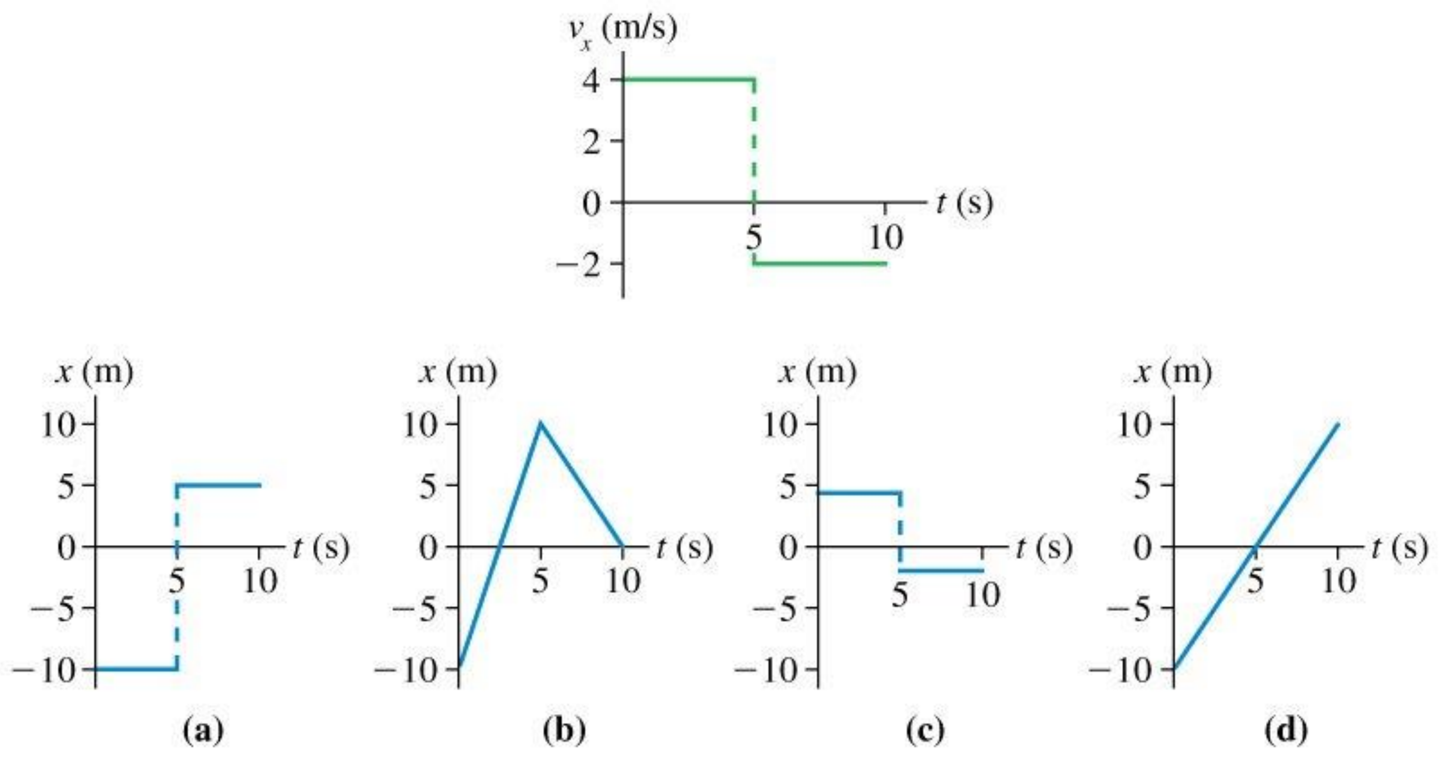
This is the same expression we had before!

Example

$$v(t) = 3t \quad \rightarrow \quad x(t) = \int 3t\, dt = \frac{3}{2}t^2 + C$$

Knowledge Inventory

- Which position vs. time graph best relates to the following velocity graph?



(b)

Another important concept to help describe non-uniform motion…

- Acceleration

  the rate of change of the velocity

  - Average (1-D)

$$\left(a_x\right)_{avg} = \frac{\Delta v_x}{\Delta t} = \frac{\left(v_x\right)_f - \left(v_x\right)_i}{t_f - t_i}$$

  - Instantaneous

$$a(t) = \frac{dv}{dt}$$

- SI Units of Acceleration:

$$\frac{m/s}{s} \quad or \quad \frac{m}{s^2}$$

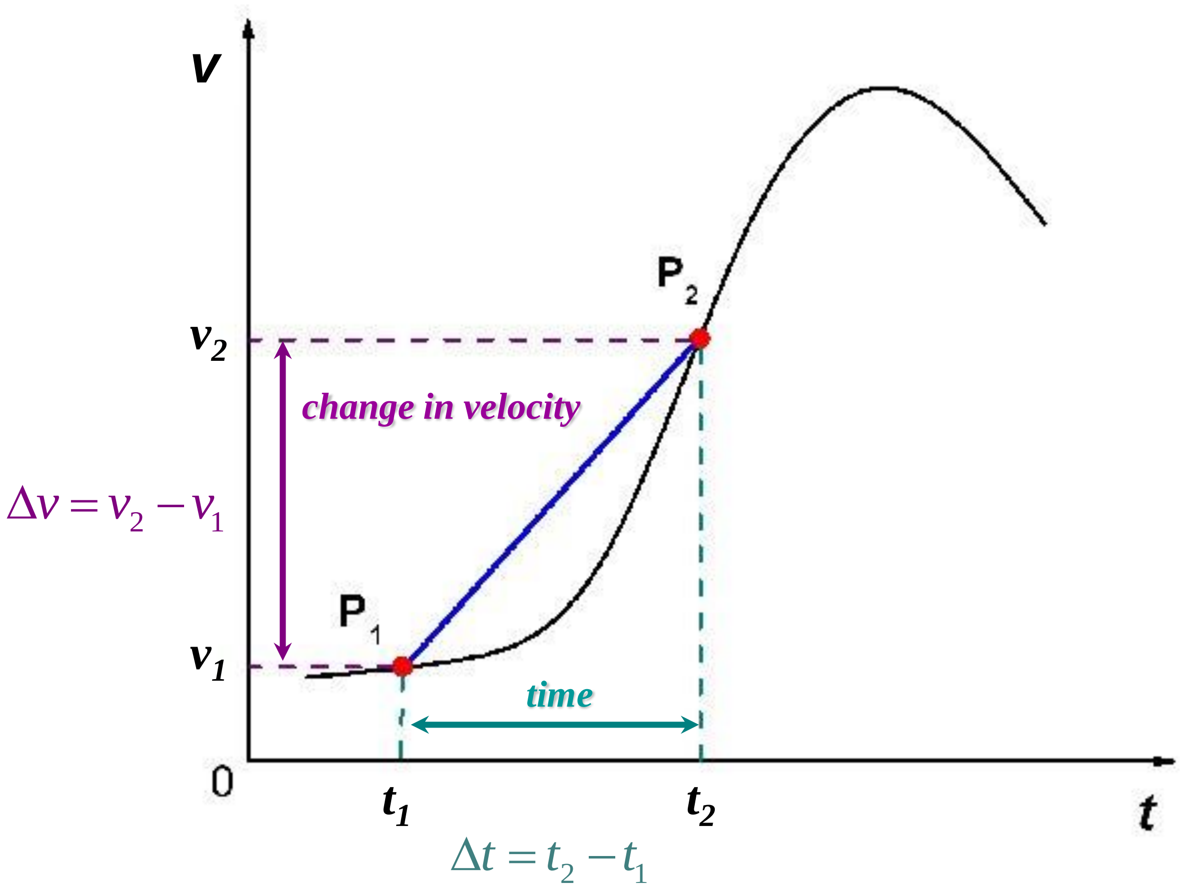
sometimes written: $m/s/s$

Ways to accelerate an object

- Since velocity is a vector, it has a magnitude **AND** a direction.

- An acceleration can occur if:
  - The magnitude only changes (linear)
  - The direction only changes (circular)
  - Both magnitude and direction change (complex)

Acceleration from a velocity graph



On a velocity vs. time graph, the ***average acceleration*** would be equal to the slope of the ***blue line*** through points P_1 & P_2.

$$slope = \frac{\Delta v}{\Delta t} \equiv a_{avg}$$

Instantaneous acceleration at time *t* is the slope of the line that is tangent to the curve on a velocity graph **AT** time *t* and is found using:

$$a = \frac{dv}{dt}$$

Additional comments regarding acceleration

- As with velocity, we will **drop** the *avg* subscript for linear motion and refer to an *average acceleration* simply as a_x or a_y.

- Alternate method for calculating acceleration:

$$a(t) = \frac{dv}{dt} \quad + \quad v(t) = \frac{dx}{dt} \quad = \quad a(t) = \frac{d^2x}{dt^2}$$

- **CAUTION**: It may take time to fully understand/grasp the concept of acceleration

Warning:

- The sign of the acceleration may **NOT** always directly indicate whether an objects velocity is increasing or decreasing.
  - ➔ Must compare signs of both **v** and **a**

- *An object is speeding up <u>if and only</u> if* **v** *and* **a** *point in the same direction*
- *An object is slowing down <u>if and only</u> if* **v** *and* **a** *point in opposite directions*
- *Velocity is constant <u>if and only</u> if* **a** *= 0*

Classroom Exercise

- The position of a car along a straight track is describe by the function:

$$x(t) = 2.1t^2 + 2.8 \ \ m$$

  - Find the displacement from t = 3 to 5 sec.
  - Find the average velocity over this time interval
  - Find the instantaneous velocity at t = 5 sec.
  - Find the average acceleration during this time
  - Find the instantaneous acceleration at t = 5 sec.

Summary

- In Non-Uniform motion, the velocity of the object is constantly changing.
- The magnitude of the velocity vector at a specific point in time is called the **instantaneous velocity**.
- Acceleration is the rate of change of the velocity.
- The sign of the acceleration may **NOT** always directly indicate whether an objects velocity is increasing or decreasing. Must compare signs of both **v** and **a**.

The Lord is not slow in keeping his promise, as some understand slowness. He is patient with you, not wanting anyone to perish, but everyone to come to repentance.

2 Peter 3:9