Vectors

[The vector] "has never been of the slightest use to any creature."

- Sir William Thomson (Lord Kelvin)

Every day objects that refer to or indicate direction.

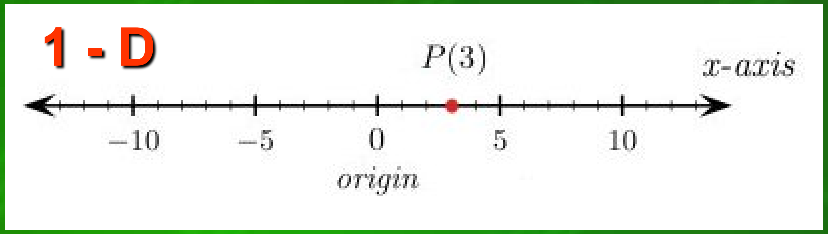
Vectors ARE extremely useful, but only if we have a uniform and consistent way to represent them.

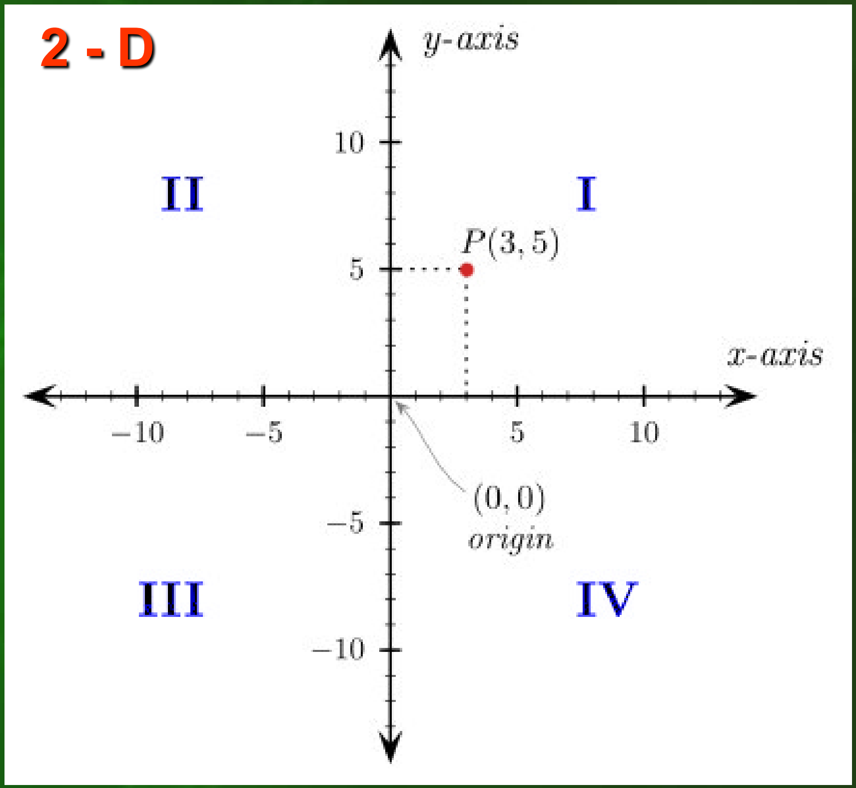
Most common types of coordinate systems:

- Cartesian
- Cylindrical
- Spherical
- Polar

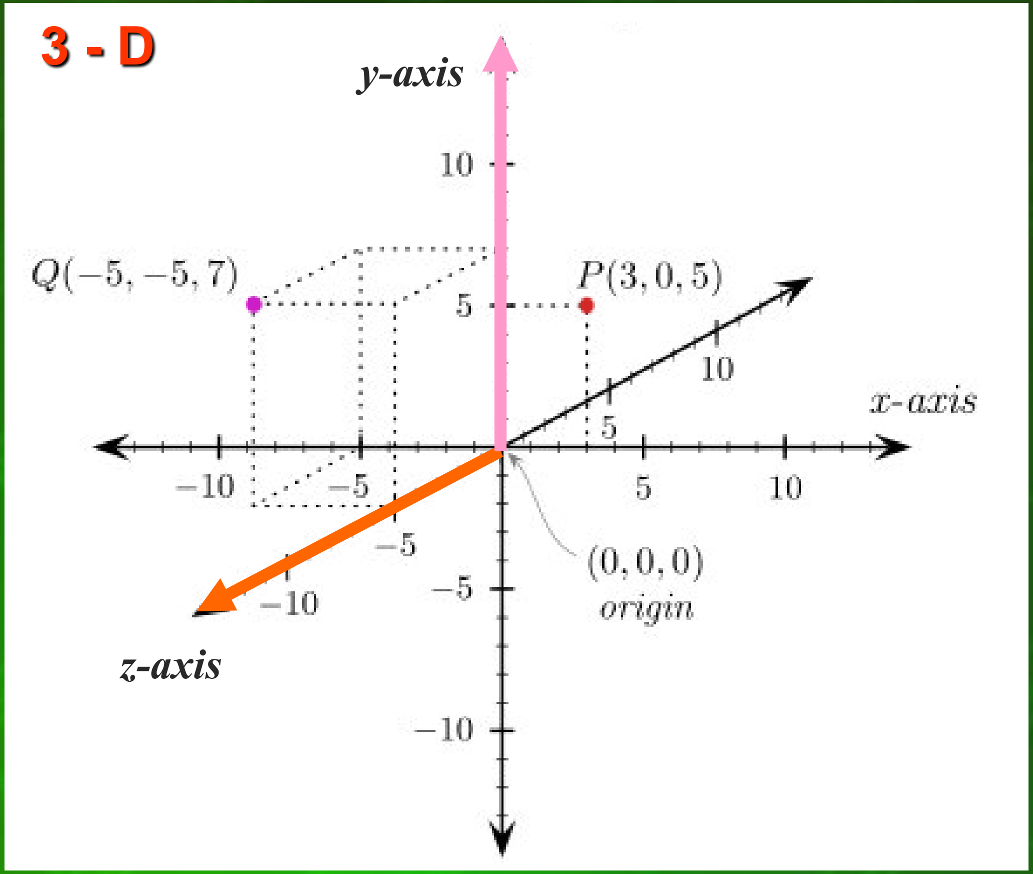
Cartesian Coordinate System

System most familiar to students. It can represent the location or spatial position of objects in 1, 2 or 3 dimensions.





Notice: Point **P** is specified by the ordered pair (*x*,*y*) in 2 - D.

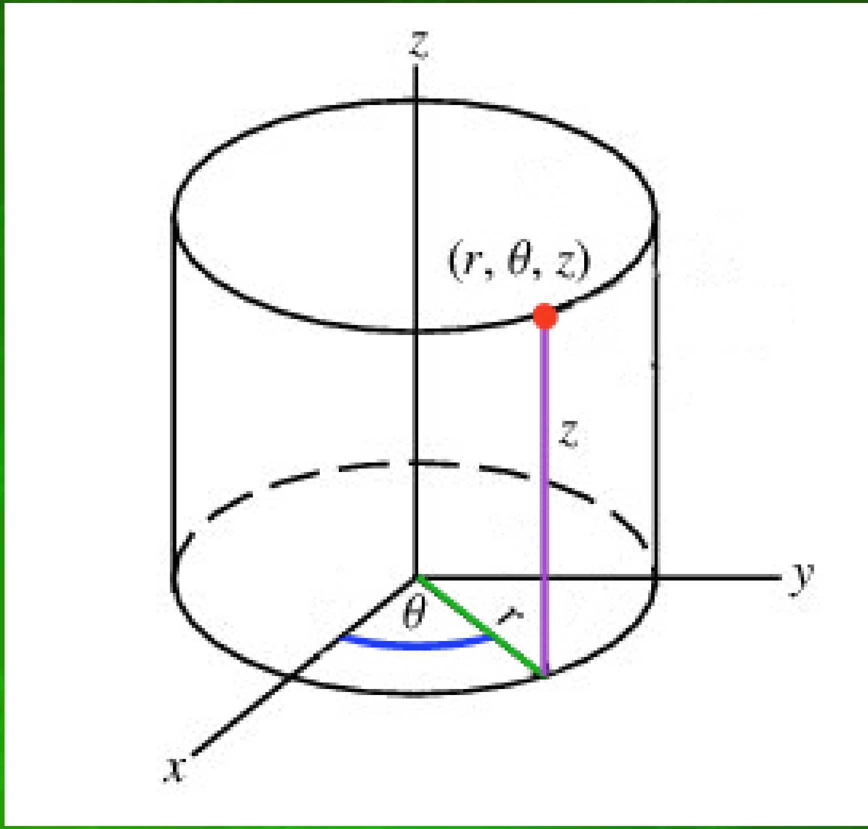


Notice: Point **P** is specified by the ordered pair (*x*,*y*,*z*).

NOTE: Sometimes the 3-D Cartesian coordinates can be drawn with the *y-axis* up and the *z-axis* outward.

Cartesian coordinates are useful for any object that has a "Box" like symmetry.

Cylindrical Coordinate System

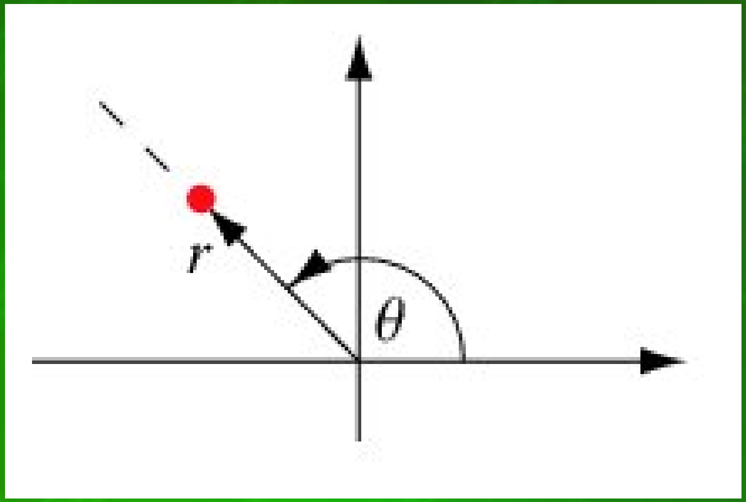


Notice: Point **P** is specified by the ordered pair (r, θ, z).

Cylindrical coordinates are useful for any object that has a “Coke can” like symmetry.

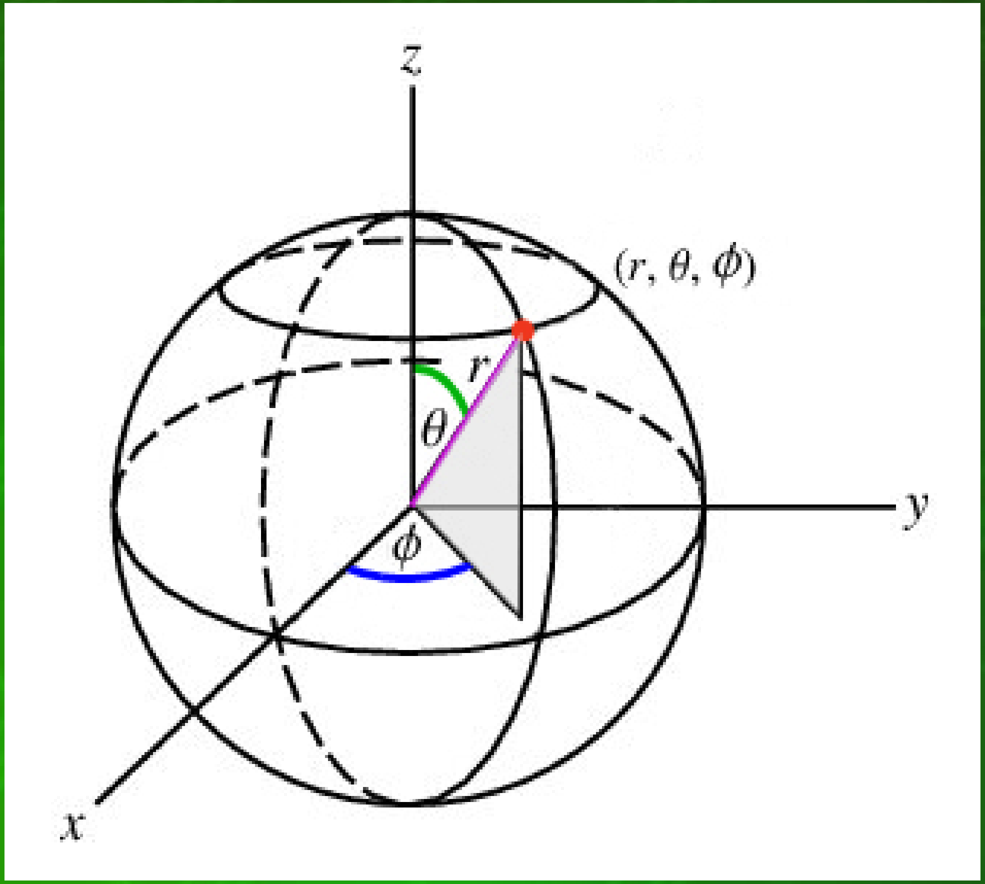
Polar Coordinate System

Polar coordinates are the 2-D form of cylindrical coordinates.



Notice: Point **P** is specified by the ordered pair (r, θ).

Spherical Coordinate System



Notice: Point **P** is specified by the ordered pair (r, θ, ϕ).

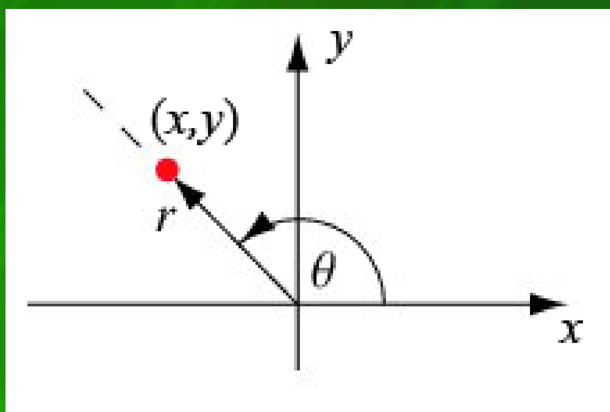
WARNING!

Calculus textbooks reverse how θ and ϕ are labeled. Thus, when using Spherical coordinates, make sure to label your axes carefully to avoid confusion.

Spherical coordinates are useful for any object that has a "Basketball" like symmetry.

Converting between coordinate systems

Ex. 2-D Cartesian and Polar coordinates.



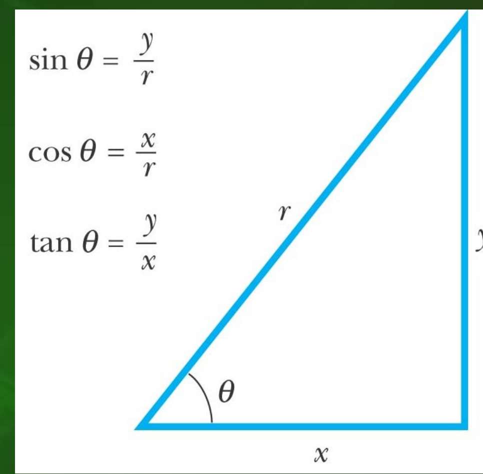
Polar → Cartesian

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Cartesian → Polar

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

These are found from the trig relations:



Knowledge Inventory

Given | **r** | = 10 *cm* and θ = 20º, find the Cartesian equivalent *x*- and *y*-components.

$$x = r\cos\theta$$
$$x = (10\,cm)\cos 20^o$$
$$x \approx 9.4\ cm$$

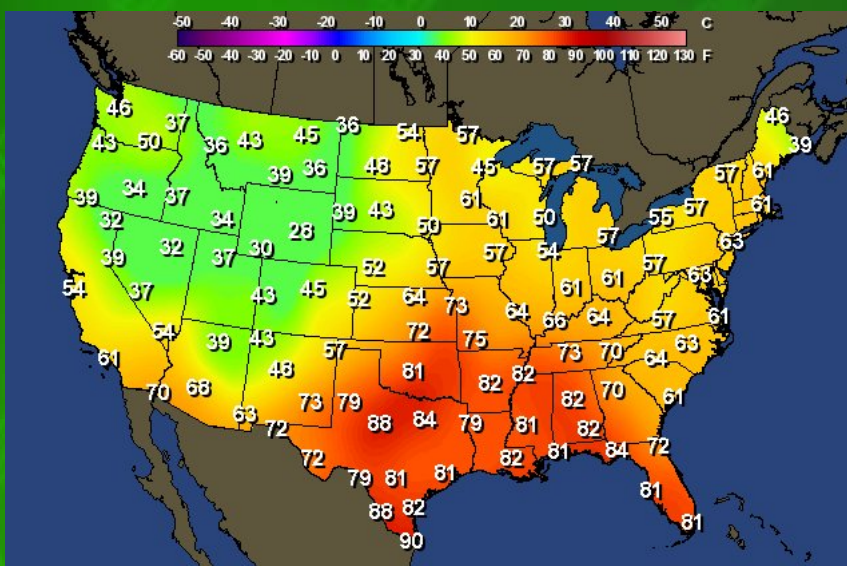
$$y = r\sin\theta$$
$$y = (10\,cm)\sin 20^o$$
$$y \approx 3.4\ cm$$

Vectors & Scalars

Scalar – a quantity with magnitude only (*number and units*)

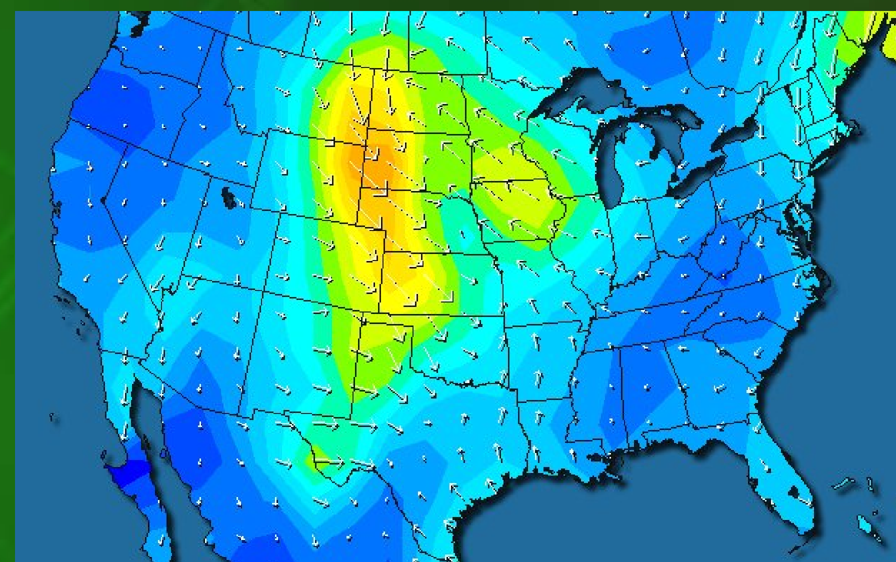
Vector – a quantity with magnitude and direction

Scalars



Temperatures

Vectors



Wind Speed & Direction

Unit Vectors

The most common way to represent vectors is by using **unit vectors**.

unit vector – a dimensionless vector with a magnitude of 1 that always points in a specific direction

Unit vectors are denoted by lower case bold letters which indicate the direction it points along with a carrot symbol over the top.

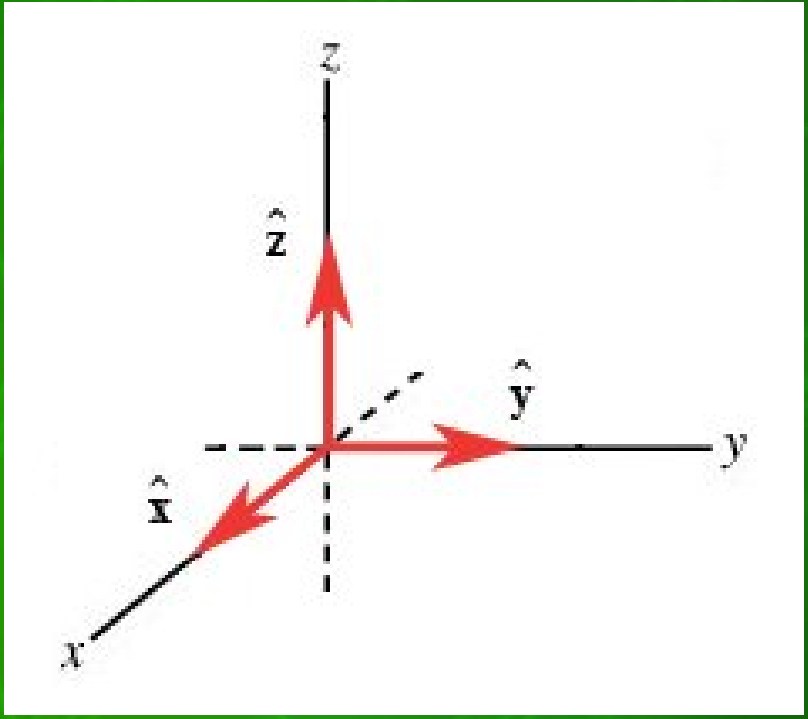
$$\hat{\mathbf{i}},\ \hat{\mathbf{e}}_{?},\ \hat{\mathbf{x}}$$

Cartesian Unit Vectors

$\hat{\mathbf{x}}\ (-\hat{\mathbf{x}})$ points down the + (–) x-axis

$\hat{\mathbf{y}}\ (-\hat{\mathbf{y}})$ points down the + (–) y-axis

$\hat{\mathbf{z}}\ (-\hat{\mathbf{z}})$ points down the + (–) z-axis

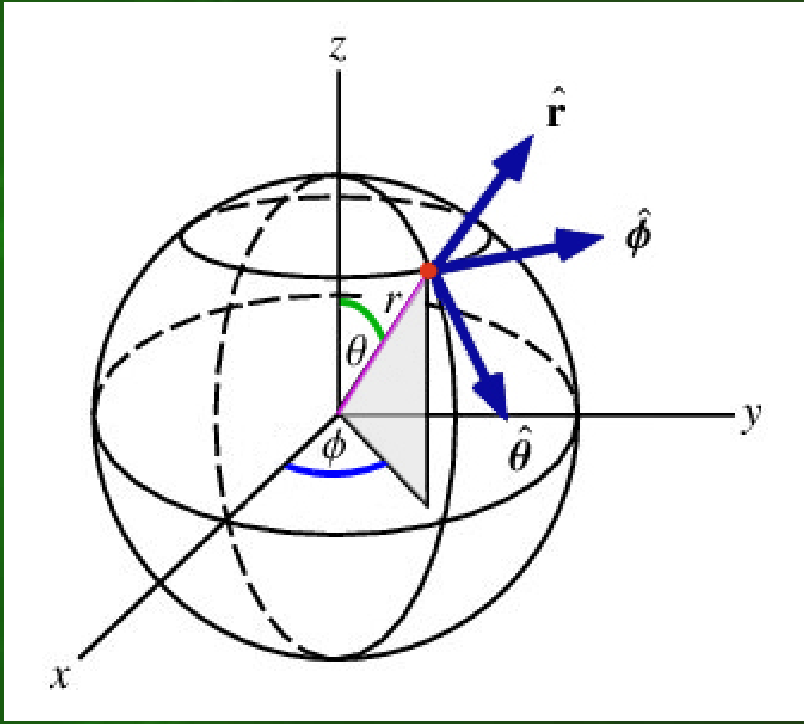


Spherical Unit Vectors

$\hat{\mathbf{r}}\ (-\hat{\mathbf{r}})$ points *away* (*toward*) the origin

$\hat{\boldsymbol{\theta}}\ (-\hat{\boldsymbol{\theta}})$

$\hat{\boldsymbol{\phi}}\ (-\hat{\boldsymbol{\phi}})$



Vector components - Cartesian

- Unit vectors allow the breakdown any vector into its components along each independent axis.

Vector Component Form

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

or

$$\mathbf{A} = \left\langle A_x, A_y, A_z \right\rangle$$

*A_x is the **x-component** of **A** and represents how much of the vector **A** lies in the x direction.*

*A_y is the **y-component** of **A** and represents how much of the vector **A** lies in the y direction.*

*A_z is the **z-component** of **A** and represents how much of the vector **A** lies in the z direction.*

Recall that you can only add vector components that are in the same or opposite direction!

Ex.

$$\mathbf{A} = \langle 6, 3 \rangle = 6\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$$
$$\mathbf{B} = \langle -2, 4 \rangle = -2\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$$
$$\mathbf{C} = \langle 1, -1 \rangle = \hat{\mathbf{x}} - \hat{\mathbf{y}}$$

Find: **A + B**
A - C
2A + C - 3B

A + B

$$= \langle 6 + (-2), 3 + 4 \rangle$$
$$= \langle 4, 7 \rangle$$

or

$$= \left(6 + (-2)\right)\hat{\mathbf{x}} + (3 + 4)\hat{\mathbf{y}}$$
$$= 4\hat{\mathbf{x}} + 7\hat{\mathbf{y}}$$

A - C

$$= \langle 6 - 1, 3 - (-1) \rangle$$
$$= \langle 5, 4 \rangle$$

or

$$= (6 - 1)\hat{\mathbf{x}} + \left(3 - (-1)\right)\hat{\mathbf{y}}$$
$$= 5\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$$

2A + C - 3B

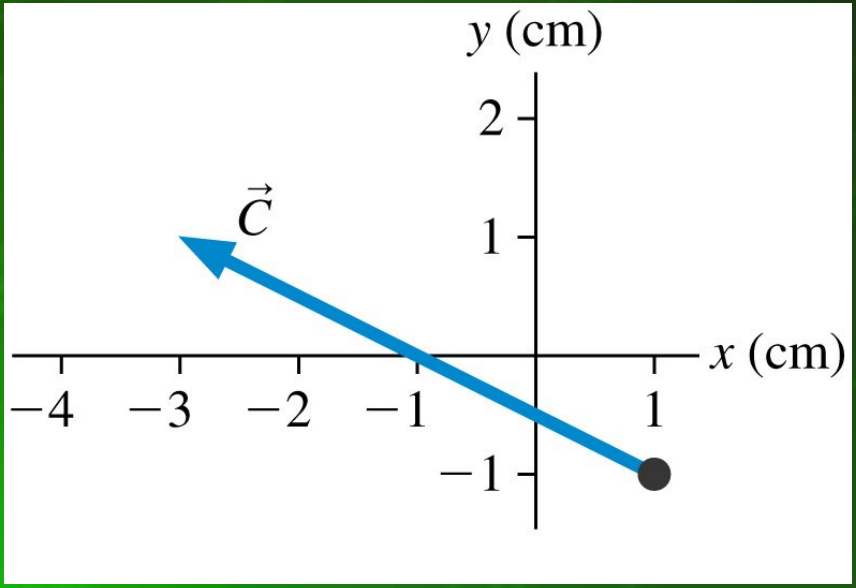
$$= \langle 2(6) + 1 - 3(-2), 2(3) + (-1) - 3(4) \rangle$$
$$= \langle 19, -7 \rangle$$

or

$$= \left(2(6) + 1 - 3(-2)\right)\hat{\mathbf{x}} + \left(2(3) + (-1) - 3(4)\right)\hat{\mathbf{y}}$$
$$= 19\hat{\mathbf{x}} - 7\hat{\mathbf{y}}$$

Knowledge Inventory

What are the Cartesian vector components of **C**?



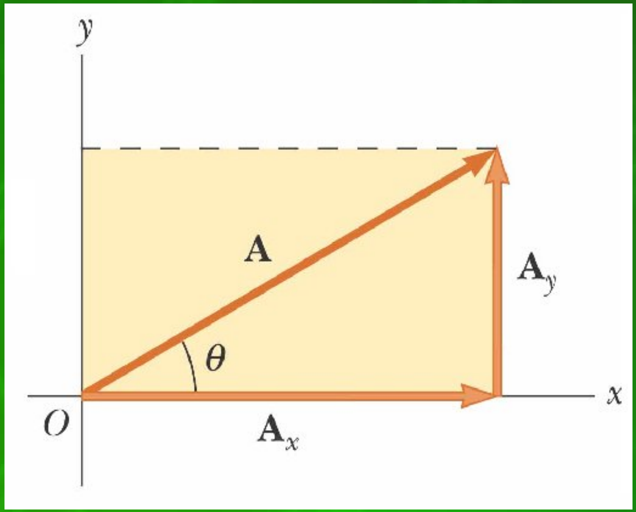
$$C_x = -4$$
$$C_y = 2$$

$$\mathbf{C} = -4\hat{\mathbf{x}} + 2\hat{\mathbf{y}} \quad or \quad \langle -4, 2 \rangle$$

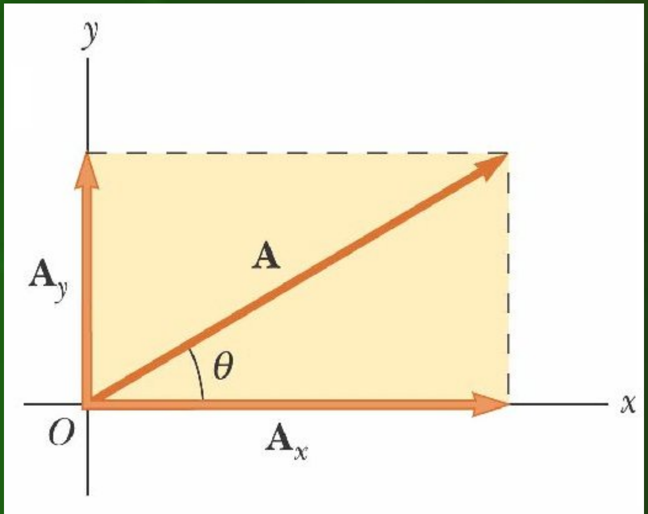
Note: The components of a vector are always displayed relative to its starting location. In this case (1, -1), not the origin (0,0)

Finding Cartesian vector components from a Polar vector.

- If we are given a polar vector, we can find its Cartesian vector components using standard trigonometry.



NOTE: The components of vector A can also be drawn as:



Given:

$|\mathbf{A}|$ *and* θ

$$A_x = |\mathbf{A}|\cos\theta$$
$$A_y = |\mathbf{A}|\sin\theta$$

Vector components - Polar

Vector Component Form

$$\mathbf{A} = r\,\hat{\mathbf{r}} + \theta\,\hat{\boldsymbol{\theta}}$$

or

$$\mathbf{A} = |\mathbf{A}| \;@\; \theta$$

or

$$\mathbf{A} = \langle |\mathbf{A}|, \theta \rangle$$

Most common polar form

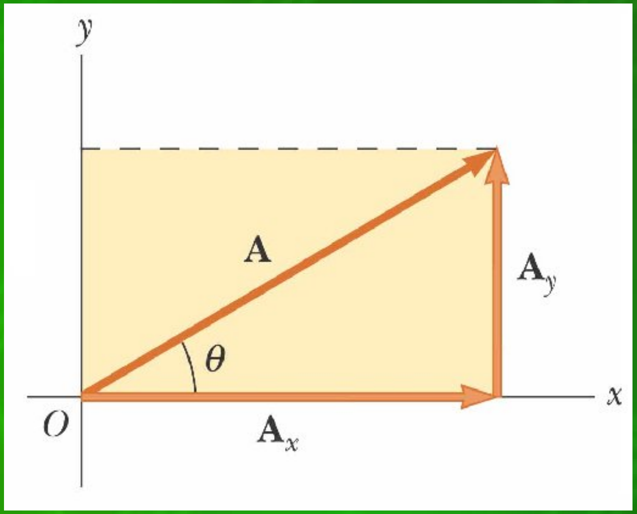
*|A| is the **magnitude** of **A** and represents the length of the vector **A** in the radial direction.*

*θ is the **direction** of **A** relative to a fixed reference line (typically the x-axis, but not always).*

NOTE: 2-D vectors are typically expressed in Cartesian (*components*) or Polar form. The choice of which to use is often a matter of preference, but at times, one form can be more useful than the other.

Finding Polar vector components from a Cartesian vector.

- If we are given a Cartesian vector, we can find the equivalent Polar vector components using standard trigonometry.



Given:

$$\mathbf{A} \;\; or \;\; A_x \;\&\; A_y$$

$$|\mathbf{A}| = \sqrt{A_x^{\;2} + A_y^{\;2}}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

Remember: The magnitude of a unit vector is 1 (unity)

$$|\hat{\mathbf{x}}| = |\hat{\mathbf{y}}| = |\hat{\mathbf{z}}| = 1$$

Ex.

$$\mathbf{A} = \langle 6, 3 \rangle = 6\hat{\mathbf{x}} + 3\hat{\mathbf{y}}$$
$$\mathbf{B} = \langle -2, 4 \rangle = -2\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$$
$$\mathbf{C} = \langle 1, -1 \rangle = \hat{\mathbf{x}} - \hat{\mathbf{y}}$$

Find: **A + B**

A - C

2A + C - 3B **<u>in Polar Coordinates</u>**

Previously, we found:

A + B

$$= 4\hat{\mathbf{x}} + 7\hat{\mathbf{y}}$$

A - C

$$= 5\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$$

2A + C - 3B

$$= 19\hat{\mathbf{x}} - 7\hat{\mathbf{y}}$$

In Polar Form:

A + B

$$r = \sqrt{4^2 + 7^2} = 8.06$$
$$\theta = \tan^{-1}\left(\frac{7}{4}\right) = 60.3^\circ$$

A - C

$$r = \sqrt{5^2 + 4^2} = 6.4$$
$$\theta = \tan^{-1}\left(\frac{4}{5}\right) = 38.7^\circ$$

2A + C - 3B

$$r = \sqrt{19^2 + (-7)^2} = 20.25$$
$$\theta = \tan^{-1}\left(\frac{-7}{19}\right) = -20.22^\circ \quad or \quad 339.78^\circ$$

Pitfall Prevention

- Be careful when calculating θ using the inverse tangent function. Generally, the inverse tangent function returns angular values between -90° and +90°. This is important to realize in the event the vector lies in the 2nd or 3rd quadrants. In those cases, the angle returned will be off by 180°.

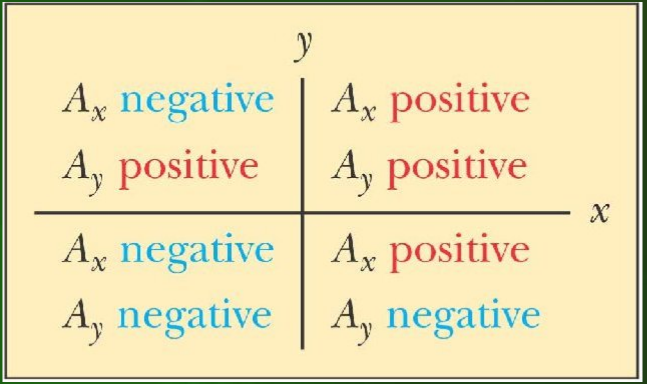
Ex.

$$\mathbf{A} = -3\hat{\mathbf{x}} + 6\hat{\mathbf{y}}$$

2nd Quadrant

$$\theta = \tan^{-1}\left(\frac{6}{-3}\right) = -63.4^{\circ}$$

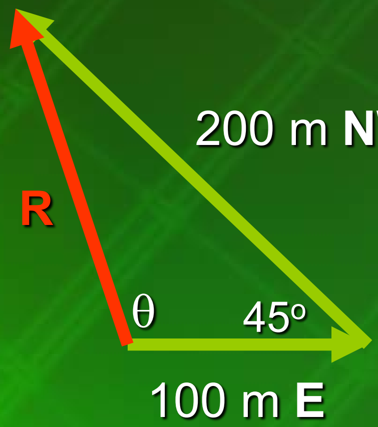
4th Quadrant
Incorrect!



$$-63.4^{\circ} + 180^{\circ} = 116.6^{\circ}$$

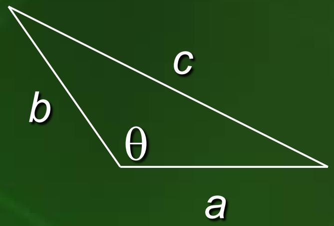
2nd Quadrant
Correct!

Ex. A car drives E for 100 *m* and then 200 *m* NW. Find the resultant vector relative to the starting point.



Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



Use the law of cosines to find R:

$$R^2 = 100^2 + 200^2 - 2(100)(200)\cos(45^\circ)$$
$$R = 147\,m$$

Use the law of cosines again to find θ:

$$200^2 = 147^2 + 100^2 - 2(100)(147)\cos\theta$$
$$\cos\theta = -0.2854$$
$$\theta = 106.6^\circ$$

Possible ways to write R:

$$\mathbf{R} = 147\,m \quad @ \quad 106.6^\circ$$
$$\mathbf{R} = 147m,\ 16.6^\circ \ \ \mathrm{W}\ of\ \mathrm{N}$$
$$\mathbf{R} = 147m,\ 73.4^\circ \ \ \mathrm{N}\ of\ \mathrm{W}$$
$$\mathbf{R}^* = 42\,m\ \mathrm{W} + 140.9\,m\ \mathrm{N}$$

** most useful form*

Vectors can be added graphically without knowing the values of the components.

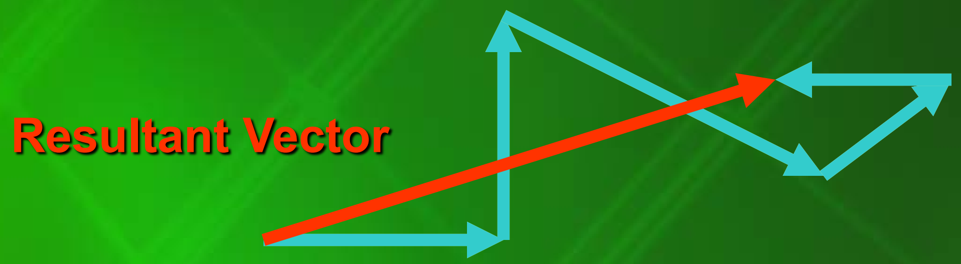
Recall:

Vectors can be represented graphically by using arrows to indicate magnitude and direction

To add vectors graphically, we use what is known as the **"Tail to Tip" Method**.

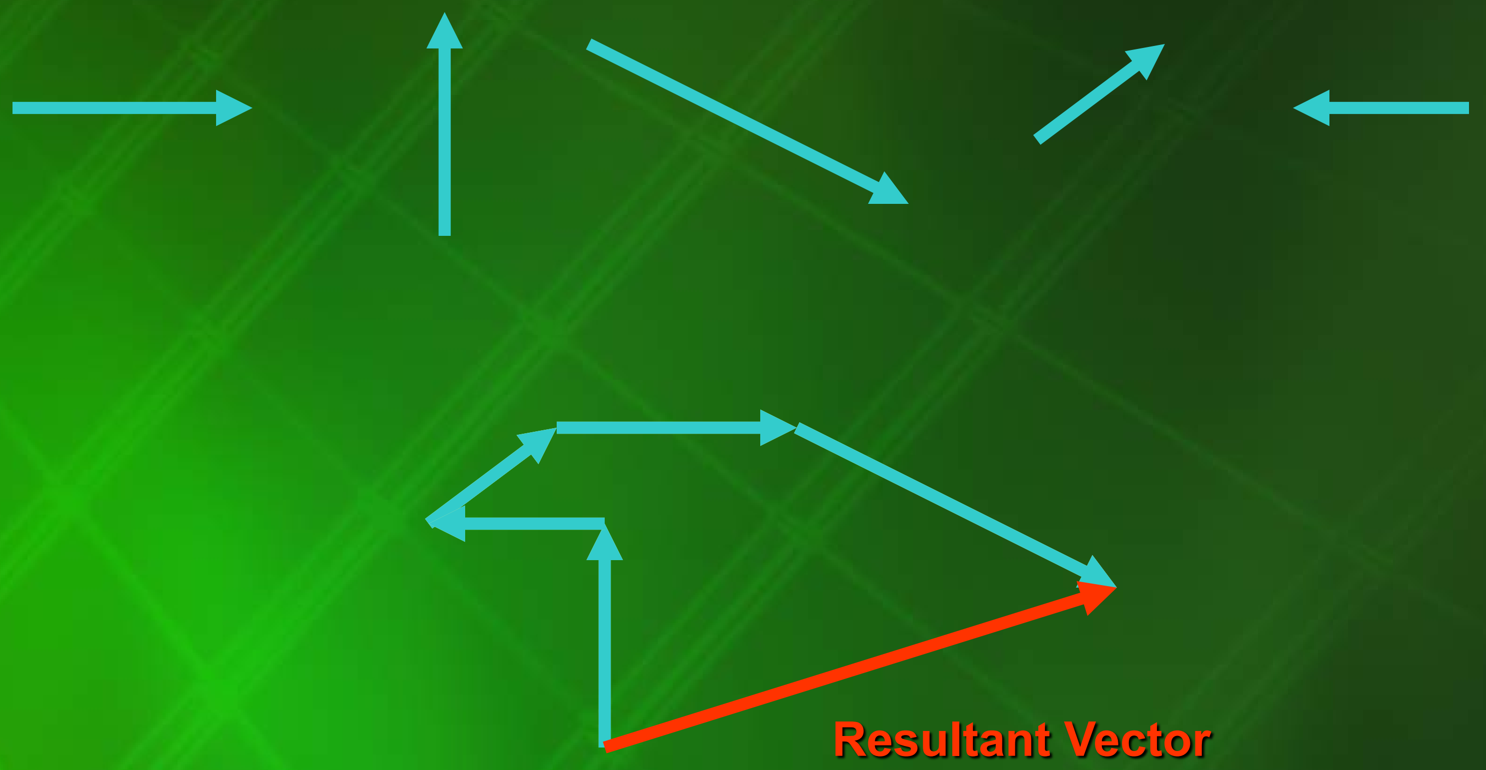
1. Start by placing any vector with its tail at the starting point (*origin*)
2. Place the tail of an unused vector at the tip of the previous vector as if the tip was the origin (*make sure to maintain the proper length and direction*)
3. After placing all the vectors in this tail to tip fashion, the **resultant vector** (*sum of all the vectors*) is found by drawing a straight line from the tail of the first vector to the tip of the last vector.

Ex. Add the following vectors graphically



NOTE: The order in which you add the vectors does **NOT** matter!

Ex. Add the following vectors graphically again using a different order

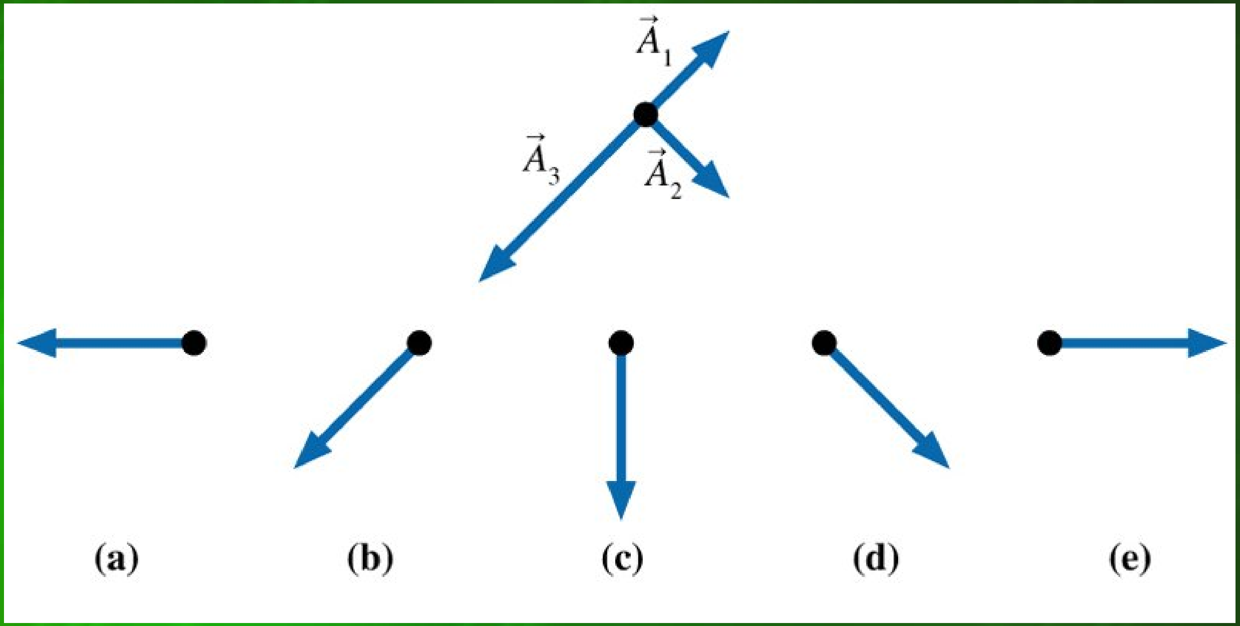


This resultant vector has exactly the same magnitude and direction as before.

Knowledge inventory

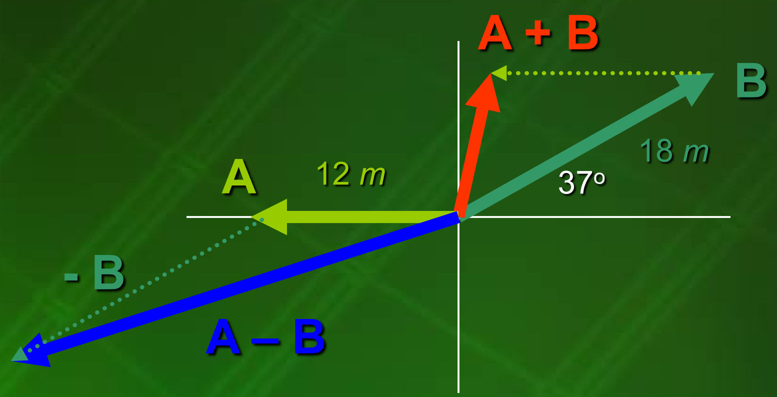
Which of the following choices shows the resultant vector for:

$A_1 + A_2 + A_3$?



(c)

Ex.



Find:

(a) **A + B**

(b) **|A + B|** & θ

(c) **A – B**

(d) **|A – B|** & θ

Need vector components for A and B

$$\mathbf{A} = \langle -12, 0 \rangle$$

$$\mathbf{B} = \langle (18\,m)\cos 37^\circ, (18\,m)\sin 37^\circ \rangle$$

$$\mathbf{B} = \langle 14.4, 10.8 \rangle$$

(a)

$$\mathbf{A}+\mathbf{B} = \langle (-12+14.4)\,m, (0+10.8)\,m \rangle$$
$$= \langle 2.4\,m, 10.8\,m \rangle$$

(c)

$$\mathbf{A}-\mathbf{B} = \langle (-12-14.4)\,m, (0-10.8)\,m \rangle$$
$$= \langle -26.4\,m, -10.8\,m \rangle$$

(b)

$$|\mathbf{A}+\mathbf{B}| = \sqrt{(2.4\,m)^2 + (10.8\,m)^2}$$
$$= 11.1\,m$$

$$\theta = \tan^{-1}\left(\frac{10.8}{2.4}\right)$$
$$= 77.5^\circ$$

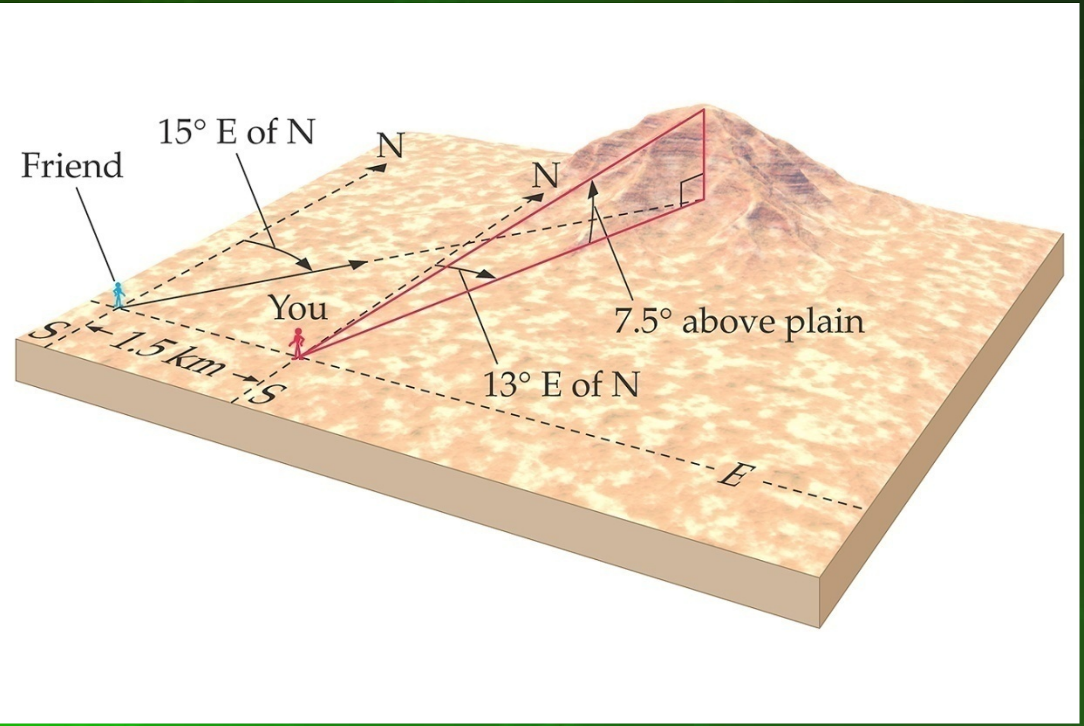
(d)

$$|\mathbf{A}-\mathbf{B}| = \sqrt{(-26.4\,m)^2 + (-10.8\,m)^2}$$
$$= 28.5\,m$$

Note quadrant

$$\theta = \tan^{-1}\left(\frac{-10.8}{-26.4}\right)$$
$$= 202.2^\circ$$

Classroom Exercise



Given the vector information in the above figure, determine the height of the mountain peak.

5.47 km

Summary

- **Coordinate Systems** help us establish a framework for indicating position and direction

- **Unit vectors** are the most common way to indicate direction and are unique to our choice of coordinate system.

- The most common and useful way to represent a vector is in **component form** (*magnitude* [scalar] + *unit vector*).

- Vector components can only be added if they are in the **same** or **opposite direction**.

- Vectors can be added graphically using the **Tail to Tip method**.

In all your ways acknowledge him, and he will make your paths straight.

Proverbs 3:6