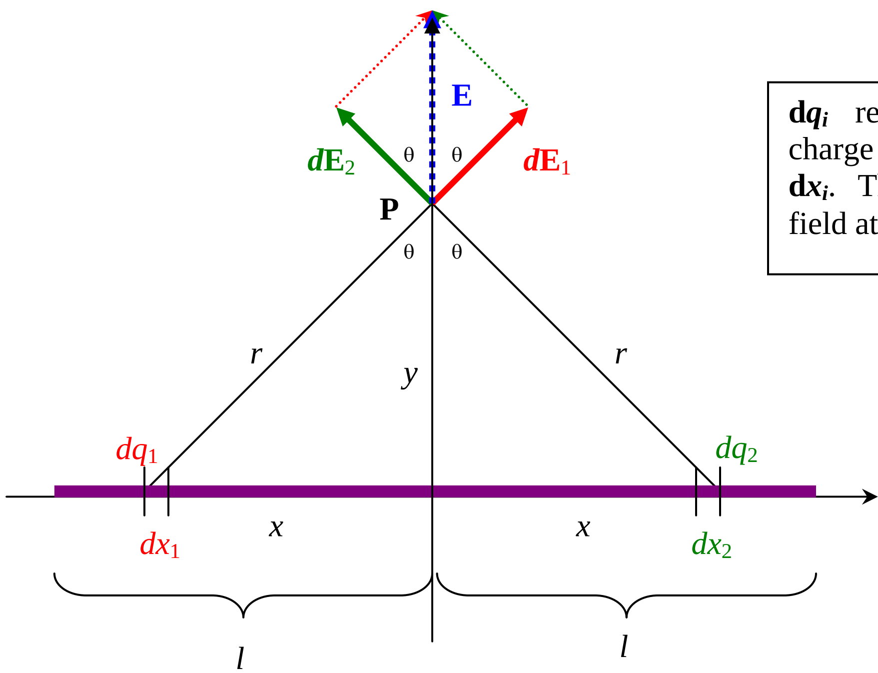
Ex. Electric Field due to a Finite Charged Rod

Find the electric field some distance y above a uniformly charged finite rod



> $\mathbf{d}\boldsymbol{q_i}$ represents the amount of charge on the rod piece of length $\mathbf{dx_i}$. The corresponding electric field at a point **P** due to $\mathbf{d}\boldsymbol{q_i}$ is $\mathbf{dE_i}$.

$$r^2 = x^2 + y^2$$

$$\cos\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sin\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

The total length of the rod is L (L = 2l).

Since this is a uniform charged rod → $\lambda = \frac{dq}{dl}$ ➔ $dq_i = \lambda dx_i$

At point P:

$$d\mathbf{E}_i = \frac{k_e|dq_i|}{r^2}\hat{\mathbf{r}} = \frac{k_e|dq_i|}{r^2}\left(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right) = \left(\frac{k_e dq_i}{r^2}\sin\theta\right)\hat{\mathbf{x}} + \left(\frac{k_e dq_i}{r^2}\cos\theta\right)\hat{\mathbf{y}}$$

The total electric field **E** due to $\boldsymbol{dq_i}$ can be found by summing up (*integrating*) all the $\boldsymbol{dq_i}$ contributions for the entire length of the rod:

$$\mathbf{E} = \int_{-l}^{l}\left(\frac{k_e dq}{r^2}\sin\theta\right)\hat{\mathbf{x}} + \left(\frac{k_e dq}{r^2}\cos\theta\right)\hat{\mathbf{y}}$$

Substituting for dq, r, sin θ and cos θ and separating the integral into its two components we get:

$$\mathbf{E} = \int_{-l}^{l}\left(\frac{k_e x\lambda dx}{\left(x^2 + y^2\right)^{3/2}}\right)\hat{\mathbf{x}} + \int_{-l}^{l}\left(\frac{k_e y\lambda dx}{\left(x^2 + y^2\right)^{3/2}}\right)\hat{\mathbf{y}}$$

The first integral ($\hat{\mathbf{x}}$) direction, is an odd function $\left[f(-x) = -f(x) \right]$. Odd functions evaluated over symmetric limits always integrate to 0.

$$\Rightarrow \qquad \mathbf{E} = 0 + \int_{-l}^{l} \left(\frac{k_e y \lambda dx}{\left(x^2 + y^2\right)^{3/2}} \right) \hat{\mathbf{y}}$$

Rearranging the limits of the integral:

$$\mathbf{E} = 2\int_{0}^{l} \left(\frac{k_e y \lambda dx}{\left(x^2 + y^2\right)^{3/2}} \right) \hat{\mathbf{y}}$$

Integrating yields:

$$\mathbf{E} = 2k_e \lambda y \left[\frac{x}{y^2 \sqrt{y^2 + x^2}} \right]\Bigg|_0^l \hat{\mathbf{y}}$$

$$\mathbf{E} = \frac{2k_e \lambda l}{y\sqrt{y^2 + l^2}} \hat{\mathbf{y}}$$

This implies that the electric field is always perpendicular to the *center of the wire*, since there is no x component in the final expression for **E**.