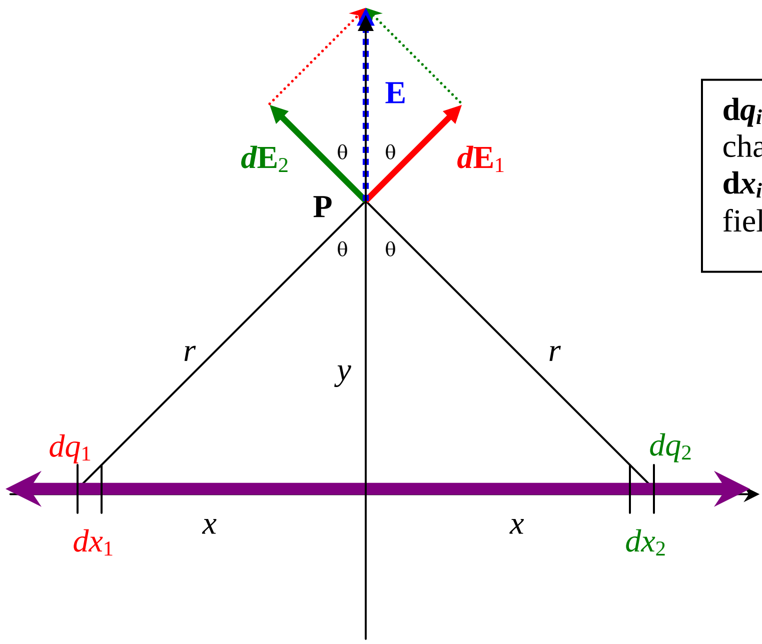
Ex. Electric Field due to an Infinite Charged Rod

Find the electric field some distance y above a uniformly charged infinite rod



dq_i represents the amount of charge on the rod piece of length **dx_i**. The corresponding electric field at a point **P** due to **dq_i** is **$d\mathbf{E}_i$**.

$$r^2 = x^2 + y^2$$

$$\cos\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sin\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

Since this is a uniform charged rod → $\lambda = \dfrac{dq}{dl}$ ➔ $dq_i = \lambda dx_i$

At point P:

$$d\mathbf{E}_i = \frac{k_e |dq_i|}{r^2}\hat{\mathbf{r}} = \frac{k_e |dq_i|}{r^2}\left(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right) = \left(\frac{k_e dq_i}{r^2}\sin\theta\right)\hat{\mathbf{x}} + \left(\frac{k_e dq_i}{r^2}\cos\theta\right)\hat{\mathbf{y}}$$

The total electric field **E** due to ***dq_i*** can be found by summing up (*integrating*) all the ***dq_i*** contributions for the entire length of the rod:

$$\mathbf{E} = \int_{-\infty}^{\infty}\left(\frac{k_e dq}{r^2}\sin\theta\right)\hat{\mathbf{x}} + \left(\frac{k_e dq}{r^2}\cos\theta\right)\hat{\mathbf{y}}$$

Substituting for dq, r, $\sin\theta$ and $\cos\theta$:

$$\mathbf{E} = \int_{-\infty}^{\infty}\left(\frac{k_e x\lambda dx}{\left(x^2 + y^2\right)^{3/2}}\right)\hat{\mathbf{x}} + \left(\frac{k_e y\lambda dx}{\left(x^2 + y^2\right)^{3/2}}\right)\hat{\mathbf{y}}$$

Since integration is distributive, I can write **E** as the sum of 2 integrals:

$$\mathbf{E} = \int_{-\infty}^{\infty}\left(\frac{k_e x\lambda dx}{\left(x^2 + y^2\right)^{3/2}}\right)\hat{\mathbf{x}} + \int_{-\infty}^{\infty}\left(\frac{k_e y\lambda dx}{\left(x^2 + y^2\right)^{3/2}}\right)\hat{\mathbf{y}}$$

The first integral ($\hat{\mathbf{x}}$) direction, is an odd function $\left[f(-x) = -f(x) \right]$. Odd functions evaluated over symmetric limits always integrate to 0.

$$\Rightarrow \qquad \mathbf{E} = 0 + \int_{-\infty}^{\infty} \left(\frac{k_e y \lambda dx}{\left(x^2 + y^2\right)^{3/2}} \right) \hat{\mathbf{y}}$$

Integrating yields:

$$\mathbf{E} = \frac{2k_e \lambda}{y} \hat{\mathbf{y}}$$

This implies that the electric field is always perpendicular to the wire at any point x along the infinite rod, since there is no x component in the final expression for **E**.

Alternate Integration:

Since the rod is symmetric about $x = 0$, we could break the integral up into 2 equal halves:

$$\mathbf{E} = \int_{-\infty}^{0} \left(\frac{k_e y \lambda dx}{\left(x^2 + y^2\right)^{3/2}} \right) \hat{\mathbf{y}} + \int_{0}^{\infty} \left(\frac{k_e y \lambda dx}{\left(x^2 + y^2\right)^{3/2}} \right) \hat{\mathbf{y}}$$

However, since the function variable being integrated represents an even function $\left[f(-x) = f(x) \right]$, the value of each of these integrals will be the same. Therefore, we can take 2 times the value of one of the integrals:

$$\mathbf{E} = 2\int_{0}^{\infty} \left(\frac{k_e y \lambda dx}{\left(x^2 + y^2\right)^{3/2}} \right) \hat{\mathbf{y}}$$

Integrating yields:

$$\mathbf{E} = \frac{2k_e \lambda}{y} \hat{\mathbf{y}}$$

This is the same result we obtained before.