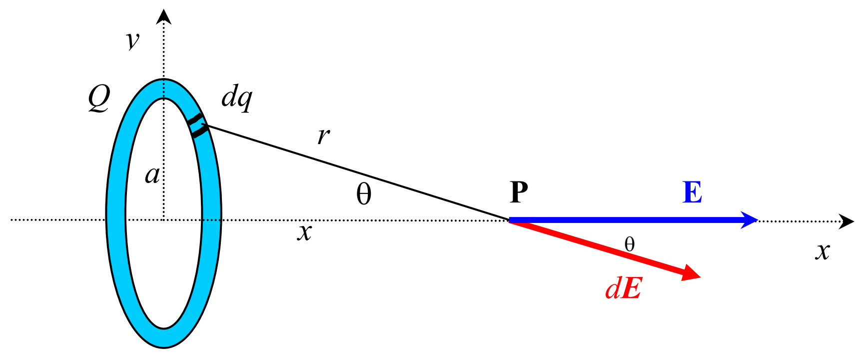
Ex. The electric field of a *Uniformly charged Ring* along the central axis



$$r = \sqrt{x^2 + a^2}$$

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + a^2}}$$

**** From Symmetry, we would expect all the *y* components to cancel out leaving only an electric field along the *x*-axis.**

METHOD I: Integrate over Charge

$$d\mathbf{E} = \left(dE\cos\theta\right)\hat{\mathbf{x}}$$

Using:

$$dE = \frac{k_e dq}{r^2} \quad \& \quad \cos\theta = \frac{x}{r},$$ we can write $d\mathbf{E}$ as:

$$d\mathbf{E} = \left(\left(\frac{k_e dq}{r^2}\right)\left(\frac{x}{\sqrt{x^2 + a^2}}\right)\right)\hat{\mathbf{x}}$$

$$d\mathbf{E} = \left(\frac{k_e x dq}{\left(x^2 + a^2\right)^{3/2}}\right)\hat{\mathbf{x}}$$

Since the Charge is uniform, we can integrate over *dq* directly holding the other variables constant:

$$\mathbf{E} = \left(\frac{k_e x\ \hat{\mathbf{x}}}{\left(x^2 + a^2\right)^{3/2}}\right)\int_0^Q dq$$

$$\mathbf{E} = \left(\frac{k_e Q x}{\left(x^2 + a^2\right)^{3/2}}\right)\hat{\mathbf{x}}$$

Limiting Cases of E:

$x = 0:$ $\mathbf{E} = 0\ \hat{\mathbf{x}}$ At this location, all the vector components for **E** cancel out.

$x >> a:$ $\mathbf{E} = \frac{k_e Q}{x^2}\ \hat{\mathbf{x}}$ Extremely far away, the result reduces to that of a point charge.

METHOD II: Integrate over the Geometry of the Ring

Using the linear charge density for a continuous object, $\lambda = \dfrac{dq}{dl}$, we get

$$dq = \lambda \, dl$$

The electric field $d\mathbf{E}$ due to dq is given by

$$d\mathbf{E} = \left(dE\cos\theta\right)\hat{\mathbf{x}}$$

$$d\mathbf{E} = \left(\left(\frac{k_e\lambda \, dl}{r^2}\right)\cos\theta\right)\hat{\mathbf{x}}$$

Integrating over the circumference of the ring yields:

$$\mathbf{E} = \left(\left(\frac{k_e\lambda \, \hat{\mathbf{x}}}{r^2}\right)\cos\theta\right)\int_{Ring} dl$$

$$\mathbf{E} = \left(\left(\frac{k_e\lambda \, \hat{\mathbf{x}}}{r^2}\right)\cos\theta\right)\left(2\pi a\right)$$

Substituting for $\cos\theta$ and r:

$$\mathbf{E} = \left(\left(\frac{k_e\lambda}{\left(x^2+a^2\right)}\right)\left(\frac{x}{\sqrt{x^2+a^2}}\right)\right)\left(2\pi a\right)\hat{\mathbf{x}}$$

$$\mathbf{E} = \left(\frac{2\pi a k_e \lambda x}{\left(x^2+a^2\right)^{3/2}}\right)\hat{\mathbf{x}}$$

Substituting in for the *linear mass density* for the whole ring, $\lambda = \dfrac{Q}{2\pi a}$, we get:

$$\mathbf{E} = \left(\frac{k_e Q x}{\left(x^2+a^2\right)^{3/2}}\right)\hat{\mathbf{x}}$$

This is the same expression for **E** we found earlier in **Method I**.