

Matrices (Part I)

Matrix – a square or rectangular array of numbers or functions that obey certain operational rules

Matrices can be denoted in 3 possible ways:

- 1) Using uppercase letters

A, B, C...

- 2) Using representative elements enclosed in brackets

$[a_{ij}], [b_{ij}], [c_{ij}]...$ where $i = 1...m$ & $j = 1...n$

m – maximum number of rows

n – maximum number of columns

- 3) Using a visual square or rectangular display

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

NOTE: Each element within the matrix has a unique label or arrangement of i & j .

Properties of Matrices:

Order:

The order of a matrix is indicated by listing the number of rows followed by the number of columns.

$$\text{order} = m \times n$$

Equality:

2 matrices are equal *iff* they are the same order and the corresponding elements in each matrix are the same.

$$a_{ij} = b_{ij} \quad \text{where } i = 1...m \text{ \& } j = 1...n$$

Matrix Addition:

Matrices can only be added if they are the same order ($m \times n$). When adding, you sum like components to create the new component in the resultant matrix.

$$A + B = [a_{ij} + b_{ij}] = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Ex.

$$\begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 9 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 7 \\ 8 & 5 \end{bmatrix}$$

Zero Matrix:

A matrix whose every entry is 0, denoted O.

Properties:

$$A + O = A$$

$$A + (-A) = O$$

If $cA = O$, then $c = 0$ or $A = O$

Transpose of a Matrix:

If

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

then the transpose of A looks like

$$A^T \text{ or } \tilde{A} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

and can be thought of as something like a mirror image.

NOTE: If A is a matrix of order $m \times n$, the A transpose is a matrix of order $n \times m$.

Ex. Find the transpose of the following matrices.

$$A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$A^T = [2 \quad 8]$$

$$B = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 3 & 9 \\ 7 & 5 & 6 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 3 & 5 \\ 4 & 9 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 3 & -5 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 4 & -5 \end{bmatrix}$$

Properties Involving Transpose Matrices:

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(cA)^T = c(A^T)$$

$$(AB)^T = B^T A^T$$

Ex. Insert example(s)

Diagonal Matrix:

A $n \times n$ (*square*) matrix in which the only non-zero elements are along the diagonal.

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Identity Matrix:

A diagonal matrix in which every diagonal element is 1.

$$I_n \text{ or } 1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad n = \text{order of square matrix}$$

Property:

$$1A = I_n A = A$$

Trace:

The trace of a matrix is the sum of all the diagonal elements.

*This applies only to $n \times n$ (*square*) matrices.*

Properties:

$$Tr(A \pm B) = Tr(A) \pm Tr(B)$$

$$Tr(cA) = c(Tr(A))$$

$$Tr(AB) = Tr(BA) \quad \textbf{Note: This is true even though } AB \neq BA$$

Ex.

$$Tr \begin{bmatrix} 2 & 3 & 4 \\ 7 & 8 & 5 \\ 2 & 1 & 6 \end{bmatrix} = 2 + 8 + 6 = 16$$

Scalar Multiplication:

When multiplying a matrix by a scalar, each element in the matrix gets multiplied by the scalar.

$$cA = [ca_{ij}] = \begin{bmatrix} ca_{11} & ca_{12} & \dots \\ ca_{21} & ca_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Multiplication by a scalar is commutative:

$$cA = Ac$$

Matrix Subtraction

Matrix subtraction is defined as:

$$A - B = A + (-1)B = [a_{ij} - b_{ij}] = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \dots \\ a_{21} - b_{21} & a_{22} - b_{22} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Ex.

$$\begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} -6 & 3 \\ 9 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -10 & -5 \end{bmatrix}$$

$$3 \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -3 & 0 \end{bmatrix}$$

Properties of Addition and Scalar Multiplication:

$$A + B = B + A$$

Commutative

$$A + (B + C) = (A + B) + C$$

Associative

$$(cd)A = c(dA) = d(cA)$$

$$1A = A$$

$$c(A + B) = cA + cB$$

Distributive

$$(c + d)A = cA + dA$$

...