

## Matrices (Part II)

### Matrix Multiplication:

Matrix-Matrix multiplication (*also known as the inner product*)

$$AB = C \quad \text{iff} \quad A \text{ is a } n \times m \text{ matrix and } B \text{ is a } m \times p \text{ matrix.}$$

The inner product is a special mathematical process that exists only if the number of columns in the first matrix is equal to the number of rows in the second matrix. The resultant matrix C is of the order  $n \times p$ .

Ex. Find the inner product for the following matrix:

$$\begin{array}{cc} \begin{bmatrix} 2 & 3 \\ -1 & -4 \end{bmatrix} & \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\ \begin{array}{cc} 2 \times 2 & 2 \times 2 \\ \hline \end{array} & = 2 \times 2 \end{array}$$

The process for matrix-matrix multiplication is as follows:

- 1) the  $i^{\text{th}}$  elements in row of A are multiplied by the  $j^{\text{th}}$  elements in column in B
- 2) the resultant element products are summed and represent the new  $ij^{\text{th}}$  element in C

$$\begin{aligned} c_{11} &= 2(4) + 3(2) = 8 + 6 = 14 \\ c_{12} &= 2(1) + 3(-2) = 2 + -6 = -4 \\ c_{21} &= -1(4) + -4(2) = -4 + -8 = -12 \\ c_{22} &= -1(1) + -4(-2) = -1 + 8 = 7 \end{aligned}$$

Therefore, the resultant matrix C looks like:

$$C = \begin{bmatrix} 14 & -4 \\ -12 & 7 \end{bmatrix}$$

Ex. Find the inner product for the following matrix:

$$\begin{array}{cc} \begin{bmatrix} 3 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \\ \begin{array}{cc} 1 \times 2 & 2 \times 2 \\ \hline \end{array} & = 1 \times 2 \quad \text{The resultant matrix will be of order } 1 \times 2. \end{array}$$

The form of the resultant matrix is then:

$$\begin{bmatrix} c_{11} & c_{12} \end{bmatrix}$$

$$c_{11} = 3(1) + -1(2) = 3 + -2 = 1$$

$$c_{12} = 3(1) + -1(-2) = 3 + 2 = 5$$

$$\rightarrow C = \begin{bmatrix} 1 & 5 \end{bmatrix}$$

*Ex.* Find the inner product for the following matrix:

$$\begin{bmatrix} -1 & 2 & 9 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$$

$$\begin{matrix} 2 \times 3 & \underbrace{\hspace{1cm}} & 3 \times 1 & = & 2 \times 1 \end{matrix} \quad \text{The resultant matrix will be of order } 2 \times 1.$$

The form of the resultant matrix is then:

$$\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}$$

$$c_{11} = -1(1) + 2(-5) + 9(7) = -1 + -10 + 63 = 52$$

$$c_{21} = 0(1) + 3(-5) + 4(7) = 0 + -15 + 28 = 13$$

$$\rightarrow C = \begin{bmatrix} 52 \\ 13 \end{bmatrix}$$

*Ex.* Find the inner product for the previous matrices but in reverse order:

$$\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} \begin{bmatrix} -1 & 2 & 9 \\ 0 & 3 & 4 \end{bmatrix}$$

$$\begin{matrix} 3 \times 1 & & 2 \times 3 \\ & \underbrace{\hspace{1cm}} & \end{matrix}$$

Inner dimensions not the same, there not possible!

### Properties of Matrix Multiplication:

$$(AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

$$c(AB) = (cA)B = A(cB)$$

$$AB \neq BA$$

Not Commutative!!!

**NOTE:** Diagonal matrices **ARE** commutative as are a select collection of non-diagonal matrices (*which will be discussed later*).

*Ex.*

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 14 & -4 \\ -12 & 7 \end{bmatrix}$$

*But*

$$BA = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 6 & 14 \end{bmatrix}$$

**NOT EQUAL!!!**

### Inverse Matrix:

An inverse matrix is defined such that an  $n \times n$  matrix times its inverse yields the identity matrix of order  $n$  ( $I_n$ ).

$$AA^{-1} = A^{-1}A = I$$

**Any matrix that has an inverse** is called **invertible** (*non-singular*).

### Properties of Inverse Matrices:

$$(A^{-1})^{-1} = A$$

$$(A^{-1})^T = (A^T)^{-1}$$

$$(cA)^{-1} = \frac{1}{c}A^{-1} \quad c \neq 0$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

## Finding the inverse of a Matrix:

The traditional method is to take a matrix and adjoin it with an identity matrix and then row-reduce using Gauss-Jordan Elimination until the left side becomes the identity matrix.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{11}{8} & -\frac{7}{18} & -\frac{1}{18} \\ 0 & 1 & 0 & -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ 0 & 0 & 1 & -\frac{1}{18} & -\frac{1}{18} & \frac{5}{8} \end{bmatrix}$$

Therefore

$$A^{-1} = \begin{bmatrix} \frac{11}{8} & -\frac{7}{18} & -\frac{1}{18} \\ -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{8} \end{bmatrix}$$

Ex. Verify that  $AA^{-1} = I_3$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{11}{8} & -\frac{7}{18} & -\frac{1}{18} \\ -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{8} \end{bmatrix}$$

$$\begin{aligned} c_{11} &= 3\left(\frac{11}{8}\right) + 2\left(-\frac{7}{18}\right) + 1\left(-\frac{1}{18}\right) = 1 \\ c_{12} &= 3\left(-\frac{7}{18}\right) + 2\left(\frac{11}{18}\right) + 1\left(-\frac{1}{18}\right) = 0 \\ c_{13} &= 3\left(-\frac{1}{18}\right) + 2\left(-\frac{1}{18}\right) + 1\left(\frac{5}{8}\right) = 0 \\ c_{21} &= 2\left(\frac{11}{8}\right) + 3\left(-\frac{7}{18}\right) + 1\left(-\frac{1}{18}\right) = 0 \\ c_{22} &= 2\left(-\frac{7}{18}\right) + 3\left(\frac{11}{18}\right) + 1\left(-\frac{1}{18}\right) = 1 \\ c_{23} &= 2\left(-\frac{1}{18}\right) + 3\left(-\frac{1}{18}\right) + 1\left(\frac{5}{8}\right) = 0 \\ c_{31} &= 1\left(\frac{11}{8}\right) + 1\left(-\frac{7}{18}\right) + 4\left(-\frac{1}{18}\right) = 0 \\ c_{32} &= 1\left(-\frac{7}{18}\right) + 1\left(\frac{11}{18}\right) + 4\left(-\frac{1}{18}\right) = 0 \\ c_{33} &= 1\left(-\frac{1}{18}\right) + 1\left(-\frac{1}{18}\right) + 4\left(\frac{5}{8}\right) = 1 \end{aligned}$$

$$\rightarrow AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$